

On the initial estimate of interface forces in FETI methods

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Abstract

The Balanced Domain Decomposition (BDD) method and the Finite Element Tearing and Interconnecting (FETI) method are two commonly used non-overlapping domain decomposition methods. Due to strong theoretical and numerical similarities, these two methods are generally considered as being equivalently efficient. However, for some particular cases, such as for structures with strong heterogeneities, FETI requires a large number of iterations to compute the solution compared to BDD. In this paper, the origin of the bad efficiency of FETI in these particular cases is traced back to poor initial estimates of the interface stresses. To improve the estimation of interface forces a novel strategy for splitting interface forces between neighboring substructures is proposed. The additional computational cost incurred is not significant. This yields a new initialization for the FETI method and restores numerical efficiency which makes FETI comparable to BDD even for problems where FETI was performing poorly. Various simple test problems are presented to discuss the efficiency of the proposed strategy and to illustrate the so-obtained numerical equivalence between the BDD and FETI solvers.

Key words: domain decomposition, iterative solver, FETI, Schur complement, force splitting

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1 Introduction

Domain Decomposition methods provide a natural framework to solve engineering problems decomposed into subparts. Such problems can arise for instance because each subdomain is discretized independently, or because the subdomains represent different physical domains. Decomposed domains can also be created from an initial single domain problem in order to make efficient use of parallel computing hardware, i.e. in order to distribute the computing work on several processors.

Among the domain decomposition methods applied in engineering mechanics to solve elliptic linear problems, two similar methods have emerged in the last decade as efficient parallel computing methods: the primal and the dual Schur complement methods. More specifically, two procedures have been shown to ensure scalability and robustness: the Balanced Domain Decomposition (BDD) method [1,2] and the Finite Element Tearing and Interconnecting (FETI) method [3,4]. The BDD method is a primal procedure where preconditioned conjugate gradient iterations are applied to find the interface displacements that satisfy the interface equilibrium. In the FETI approach, it is the interface forces that are searched for iteratively so as to satisfy the interface displacement compatibility. Therefore FETI is sometimes referred to as a dual method.

The primal and dual Schur complement methods are based on very similar concepts (see e.g. [5] for a mechanical description). Mathematically, it has been shown that the preconditioned interface operators for both the BDD and FETI methods have a condition number bounded by [6,7]

$$\kappa = O\left(1 + \log\frac{H}{h}\right)^2 \quad (1)$$

where H and h represent the subdomain and the mesh size respectively. Hence it is often accepted in the Domain Decomposition community that using one method instead of the other is a matter of taste and implementation preferences. However, when looking closely at the details and variants of both methods, it becomes rapidly clear that showing the exact equivalence between the primal BDD and the dual FETI is not easy, if at all possible [8].

Let us then consider the simple example depicted in Figure 1 of a highly heterogeneous elastic cube ($\frac{E_1}{E_2} = 10^5$) subdivided into $3 \times 3 \times 3$ subdomains. A uniform pressure is applied on the face opposite to the clamped side. The structure is discretized using Q_2 hexahedral finite elements with 27 nodes. The model contains 21000 degrees of freedom, of which 6000 belong to subdomain interfaces. In Figure 2 the convergence curves of the global equilibrium residual corresponding to iterations of the BDD and the FETI methods are

plotted. Both methods are equipped with what literature refers to as the best preconditioners and coarse grids (see section 2). Although the asymptotic convergence of both procedures is similar, it is observed that the convergence of the FETI method is less monotonic and that its initial residual is significantly higher.

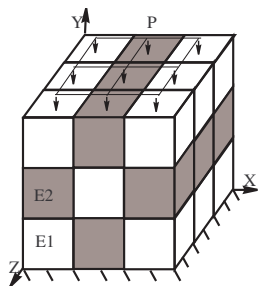


Fig. 1. Decomposed heterogeneous cube

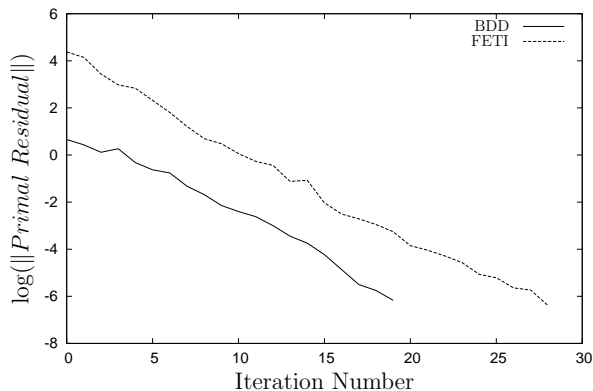


Fig. 2. Convergence of BDD & FETI

These results seem to indicate that some specific details such as the choice of the initial estimates result in possibly significant differences between the primal and dual algorithms.

In this paper we revisit the way the interface forces are estimated initially in the FETI method in an attempt to obtain a convergence comparable to that of the BDD.

In the next section, we shortly recall the concepts underlying the FETI solver. In section 3 we explain that the forces applied on the interface can be split in different ways. Although the final result is independent of that splitting, it affects the actual FETI iteration history. We then present an efficient way to define such a splitting and show how it is related to the construction of the initial iterate of FETI in section 4. Numerical examples are reported in section 5 to illustrate the effectiveness of the new initialization strategy. Finally, we present some conclusions.

2 FETI basics

2.1 The decomposed problem

Let us consider a domain Ω subdivided into N_s non-overlapping subdomains $\Omega^{(s)}$ and assume that we are solving a linear (or linearized) static equilibrium

problem on the domain. The discretized subdomain equilibrium is expressed by

$$\mathbf{K}^{(s)}\mathbf{u}^{(s)} = \mathbf{f}^{(s)} + \mathbf{g}^{(s)} \quad s = 1, \dots, N_s \quad (2)$$

where $\mathbf{K}^{(s)}$, $\mathbf{u}^{(s)}$ and $\mathbf{f}^{(s)}$ are the subdomain stiffness matrices, displacements and applied forces respectively. $\mathbf{g}^{(s)}$ are the connecting forces on the interface between subdomains (thus zero on the internal degrees of freedom). For the sake of simplicity, we assume in the following that the meshes are matching (conforming) on the interface.

The interface forces satisfy an interface equilibrium equation expressing that when assembled on the interface, the resultant is null (action-reaction):

$$\sum_{s=1}^{N_s} \mathbf{L}^{(s)T} \mathbf{g}^{(s)} = \mathbf{0} \quad (3)$$

where $\mathbf{L}^{(s)}$ is a Boolean assembly matrix. The interface connecting forces are such that the interface degrees of freedom are compatible, namely

$$\sum_{s=1}^{N_s} \mathbf{B}^{(s)} \mathbf{u}^{(s)} = \mathbf{0} \quad (4)$$

This relation expresses that for any pair $(u^{(s)}, u^{(r)})$ of degrees of freedom matching and the interface, $u^{(s)} - u^{(r)} = 0$. $\mathbf{B}^{(s)}$ are thus signed Boolean matrices expressing the compatibility constraints on the interface.

The equilibrium problem of domain Ω is fully described by the local equilibrium (2) and by the interface constraints (3, 4). In block diagonal notations, it can be summarized as

$$\begin{cases} \mathbf{K}\mathbf{u} = \mathbf{f} + \mathbf{g} \\ \mathbf{L}^T \mathbf{g} = \mathbf{0} \\ \mathbf{B}\mathbf{u} = \mathbf{0} \end{cases} \quad (5)$$

where \mathbf{K} is the block diagonal matrix of the local operators $\mathbf{K}^{(s)}$ and where

$$\begin{aligned} \mathbf{u} &= \begin{bmatrix} \mathbf{u}^{(1)} \\ \vdots \\ \mathbf{u}^{(N_s)} \end{bmatrix} & \mathbf{f} &= \begin{bmatrix} \mathbf{f}^{(1)} \\ \vdots \\ \mathbf{f}^{(N_s)} \end{bmatrix} & \mathbf{g} &= \begin{bmatrix} \mathbf{g}^{(1)} \\ \vdots \\ \mathbf{g}^{(N_s)} \end{bmatrix} \\ \mathbf{L}^T &= \begin{bmatrix} \mathbf{L}^{(1)T} & \dots & \mathbf{L}^{(N_s)T} \end{bmatrix} \\ \mathbf{B} &= \begin{bmatrix} \mathbf{B}^{(1)} & \dots & \mathbf{B}^{(N_s)} \end{bmatrix} \end{aligned} \quad (6)$$

Note that in this description, one set of interface displacements and one set of interface forces are defined per subdomain.

2.2 Solvers for decomposed problems

Solving (5) can be done in several ways:

- Considering (5) as a constrained equilibrium problem in terms of \mathbf{u} and \mathbf{g} leads to the three-field formulation of decomposed domains (see e.g. [9,10]).
- One can choose to work with a displacement set \mathbf{u} that satisfies a priori the interface compatibility (4). For that purpose we define a global set \mathbf{u}_g of degrees of freedom unique on the interface such that

$$\mathbf{u}^{(s)} = \mathbf{L}^{(s)} \mathbf{u}_g \quad \text{or} \quad \mathbf{u} = \mathbf{L} \mathbf{u}_g \quad (7)$$

where $\mathbf{L}^{(s)}$ is the same assembly Boolean matrix as in (3) that extracts subdomain degrees of freedom from the global set. Stating that $\mathbf{u}^{(s)}$ are obtained from a unique set is obviously equivalent to stating the interface compatibility (4) and (7) thus implies

$$\mathbf{B} \mathbf{u} = \mathbf{B} \mathbf{L} \mathbf{u}_g = \mathbf{0} \quad (8)$$

for any global displacement \mathbf{u}_g . On the other hand, all compatible displacements can be written as in (7). Hence

$$\mathbf{L} = \text{null}(\mathbf{B}) \quad (9)$$

In order to illustrate these concepts, we reproduce the example of [11] in Figure 3. Introducing (7) in (5) yields

$$\begin{cases} \mathbf{K} \mathbf{L} \mathbf{u}_g = \mathbf{f} + \mathbf{g} \\ \mathbf{L}^T \mathbf{g} = \mathbf{0} \end{cases} \quad (10)$$

This set of equations is at the basis of the primal iterative solution techniques such as the Primal Schur Complement or the BDD methods [1,12]: iteration schemes are applied to find the displacements \mathbf{u}_g until the interface equilibrium $\mathbf{L}^T \mathbf{g} = \mathbf{L}^T (\mathbf{K} \mathbf{L} \mathbf{u}_g - \mathbf{f}) = \mathbf{0}$ is satisfied.

- One can choose in (5) a set of interface forces satisfying a priori the interface equilibrium $\mathbf{L}^T \mathbf{g} = \mathbf{0}$ while keeping redundant interface degrees of freedom in \mathbf{u} . According to (9), such interface forces have the generic expression

$$\mathbf{g}^{(s)} = -\mathbf{B}^{(s)T} \boldsymbol{\lambda} \quad \text{or} \quad \mathbf{g} = -\mathbf{B}^T \boldsymbol{\lambda} \quad (11)$$

$\boldsymbol{\lambda}$ are interface forces that act in opposite directions between any pair of matching degrees of freedom on the interface and are therefore in equilibrium

$$\begin{array}{ccccc}
\Omega^{(1)} & & \lambda_1 & & \Omega^{(2)} \\
& u_1^{(1)} & & u_1^{(2)} & \\
& & & & \\
& u_2^{(1)} & \lambda_2 & u_2^{(2)} & \\
& & \lambda_3 & \lambda_4 & \\
& & u_1^{(3)} & & \\
& & & & \Omega^{(3)}
\end{array}$$

$$\underbrace{\begin{bmatrix} \overbrace{-1 \ 0}^{B^{(1)}} & \overbrace{1 \ 0}^{B^{(2)}} & \overbrace{0}^{B^{(3)}} \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}}_B \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}}_L \begin{bmatrix} u_{g1} \\ u_{g2} \end{bmatrix}$$

Fig. 3. Lagrange multipliers and interface compatibility

(see Figure 3). Problem (5) becomes

$$\begin{cases} \mathbf{K}\mathbf{u} + \mathbf{B}^T\boldsymbol{\lambda} = \mathbf{f} \\ \mathbf{B}\mathbf{u} = \mathbf{0} \end{cases} \quad (12)$$

Clearly, $\boldsymbol{\lambda}$ are the Lagrange multipliers associated to the interface compatibility constraints. This form of the decomposed problem is the basis for the dual procedures such as FETI: iterative algorithms are applied to compute the interface forces $\boldsymbol{\lambda}$ such that the displacements resulting from the subdomain equilibrium are compatible on the interface.

- If one chooses interface displacements that are unique on part of the interface while, on the remainder of the interface, equilibrated connecting forces are defined, one obtains hybrid primal/dual approaches such as the FETI-DP procedure [13].
- If on the entire interface we use displacements and forces that satisfy a linear

combination of the interface equilibrium and compatibility (e.g. Robin type of boundary conditions), one obtains formulations typically used in wave propagation analysis such as described for instance by Helmholtz equations [14].

- Finally if both the interface equilibrium and compatibility are enforced a priori, one obtains the fully assembled form

$$\mathbf{L}^T \mathbf{K} \mathbf{L} \mathbf{u}_g = \mathbf{L}^T \mathbf{f} \quad (13)$$

2.3 FETI: the dual iterative solver

In the FETI method [4], the decomposed problem (12) is expressed in terms of interface forces $\boldsymbol{\lambda}$: using the subdomain equilibrium equations to eliminate $\mathbf{u}^{(s)}$,

$$\mathbf{u}^{(s)} = \mathbf{K}^{(s)+} \left(\mathbf{f}^{(s)} - \mathbf{B}^{(s)T} \boldsymbol{\lambda} \right) - \mathbf{R}^{(s)} \boldsymbol{\alpha}^{(s)} \quad (14)$$

where $\mathbf{K}^{(s)+}$ is the inverse of $\mathbf{K}^{(s)}$ or a generalized inverse if subdomain $\Omega^{(s)}$ is floating when disconnected from its neighbors. In the latter case, $\mathbf{R}^{(s)}$ are the associated rigid body modes, their amplitudes $\boldsymbol{\alpha}^{(s)}$ being determined such that the interface forces are in equilibrium with the applied forces $\mathbf{f}^{(s)}$, i.e. such that the subdomain equilibrium is well-posed:

$$\mathbf{R}^{(s)T} \left(\mathbf{f}^{(s)} - \mathbf{B}^{(s)T} \boldsymbol{\lambda} \right) = \mathbf{0} \quad (15)$$

Substituting (14) into the interface compatibility condition, and taking account of (15), one obtains the dual interface problem

$$\begin{bmatrix} \mathbf{F}_I & \mathbf{G}_I \\ \mathbf{G}_I^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\alpha} \end{bmatrix} = \begin{bmatrix} \mathbf{d} \\ \mathbf{e} \end{bmatrix} \quad (16)$$

where

$$\begin{aligned}
\mathbf{F}_I &= \sum_{s=1}^{N_s} \mathbf{B}^{(s)} \mathbf{K}^{(s)+} \mathbf{B}^{(s)T} = \mathbf{B} \mathbf{K}^+ \mathbf{B}^T \\
\mathbf{d} &= \sum_{s=1}^{N_s} \mathbf{B}^{(s)} \mathbf{K}^{(s)+} \mathbf{f}^{(s)} = \mathbf{B} \mathbf{K}^+ \mathbf{f} \\
\mathbf{G}_I &= \begin{bmatrix} \mathbf{B}^{(1)} \mathbf{R}^{(1)} & \dots & \mathbf{B}^{(N_s)} \mathbf{R}^{(N_s)} \end{bmatrix} = \mathbf{B} \mathbf{R} \\
\boldsymbol{\alpha} &= \begin{bmatrix} \boldsymbol{\alpha}^{(1)} \\ \vdots \\ \boldsymbol{\alpha}^{(N_s)} \end{bmatrix} \quad \text{and} \quad \mathbf{e} = \begin{bmatrix} \mathbf{R}^{(1)T} \mathbf{f}^{(1)} \\ \vdots \\ \mathbf{R}^{(N_s)T} \mathbf{f}^{(N_s)} \end{bmatrix} = \mathbf{R}^T \mathbf{f}
\end{aligned}$$

Let us define the projection operator \mathbf{P} such that $\mathbf{G}_I^T \mathbf{P} = \mathbf{0}$:

$$\mathbf{P}(\mathbf{Q}) = \mathbf{I} - \mathbf{Q} \mathbf{G}_I \left(\mathbf{G}_I^T \mathbf{Q} \mathbf{G}_I \right)^{-1} \mathbf{G}_I^T \quad (17)$$

The choice of the operator \mathbf{Q} is discussed later. Introducing the splitting

$$\boldsymbol{\lambda} = \mathbf{P}(\mathbf{Q}) \bar{\boldsymbol{\lambda}} + \boldsymbol{\lambda}_0 \quad (18)$$

$$\boldsymbol{\lambda}_0 = \mathbf{Q} \mathbf{G}_I \left(\mathbf{G}_I^T \mathbf{Q} \mathbf{G}_I \right)^{-1} \mathbf{e} \quad (19)$$

the dual interface problem (16) is equivalent to

$$\mathbf{P}(\mathbf{Q})^T \mathbf{F}_I \mathbf{P}(\mathbf{Q}) \bar{\boldsymbol{\lambda}} = \mathbf{P}(\mathbf{Q})^T (\mathbf{d} - \mathbf{F}_I \boldsymbol{\lambda}_0) \quad (20)$$

The FETI method consists in preconditioned conjugate gradient iterations on the dual interface problem (20) [4]. Applying \mathbf{F}_I to an iterate is a naturally parallel operation. The projection steps however require solving a coarse global problem. The preconditioning operators and the possible choices for \mathbf{Q} are summarized next.

2.4 FETI preconditioners and coarse grid space

The forces $\mathbf{B}^T \boldsymbol{\lambda}$ exist only on the interface degrees of freedom. Using a subscript b and i for interface boundary and internal degrees of freedom respectively, one can re-write the decomposed problem (12) by condensing out \mathbf{u}_i :

$$\begin{cases} \mathbf{S} \mathbf{u}_b + \mathbf{B}_b^T \boldsymbol{\lambda} = \mathbf{f}_b^* \\ \mathbf{B}_b \mathbf{u}_b = \mathbf{0} \end{cases} \quad (21)$$

where \mathbf{B}_b is a submatrix of \mathbf{B} associated to boundary displacements only and

$$\mathbf{S} = \begin{bmatrix} & & \ddots & & \\ & & & & \\ & \mathbf{K}_{bb}^{(s)} - \mathbf{K}_{bi}^{(s)} \mathbf{K}_{ii}^{(s)-1} \mathbf{K}_{ib}^{(s)} & & & \\ & & & \ddots & \\ & & & & \end{bmatrix} \quad (22)$$

$$\mathbf{f}_b^* = \begin{bmatrix} \vdots \\ \mathbf{f}_b^{(s)} - \mathbf{K}_{bi}^{(s)} \mathbf{K}_{ii}^{(s)-1} \mathbf{f}_i^{(s)} \\ \vdots \end{bmatrix} \quad (23)$$

\mathbf{S} is the block diagonal matrix of the local operators statically condensed on the interface (also known as Schur complements). The dual interface problem (16) or (20) can thus be written in equivalent forms by replacing \mathbf{K} , \mathbf{f} and \mathbf{B} by \mathbf{S} , \mathbf{f}_b^* and \mathbf{B}_b respectively.

The preconditioner in FETI for which the optimal conditioning number (1) holds is the Dirichlet preconditioner:

$$\tilde{\mathbf{F}}_{I_D}^{-1} = \tilde{\mathbf{B}}_b \mathbf{S} \tilde{\mathbf{B}}_b^T \quad (24)$$

The operator $\tilde{\mathbf{B}}_b$ is similar to \mathbf{B}_b but includes a scaling with respect to interface multiplicity or relative interface stiffness [15]. The scaling is implemented as a simple pre- and post-processing in the preconditioning step. Mathematically, its general expression is [7, 11]

$$\tilde{\mathbf{B}}_b = \left(\mathbf{B}_b \mathbf{A} \mathbf{B}_b^T \right)^+ \mathbf{B}_b \mathbf{A} \quad (25)$$

where the $^+$ superscript denotes an inverse or a pseudo-inverse in case redundant compatibility constraints are present. It can be shown that the multiplicity scaling procedure corresponds to $\mathbf{A} = \mathbf{I}$, whereas the stiffness scaling (also known as super-lumped scaling) corresponds to $\mathbf{A} = \text{diag}(\mathbf{K}_{bb})^{-1}$.

The operator \mathbf{Q} in the coarse grid projector (17) defines the interface forces $\mathbf{Q}\mathbf{G}_I$ associated to rigid mode displacements of subdomains. It can be chosen as $\mathbf{Q} = \mathbf{I}$, but for many engineering problems it should be taken as the preconditioner, namely

$$\mathbf{Q} = \tilde{\mathbf{F}}_{I_D}^{-1} \quad (26)$$

or at least as a degenerated form of the preconditioner such as

$$\mathbf{Q} = \left(\mathbf{B}_b \text{diag}(\mathbf{K}_{bb})^{-1} \mathbf{B}_b^T \right)^+ \quad (27)$$

Further discussion on preconditioning, scaling and on the choice of \mathbf{Q} can be

found in [4, 11, 15, 16].

3 Splitting forces on an interface

The definition of the decomposed hybrid problem (12) is not unique: any interface force satisfying the interface equilibrium can be added to the system without changing the final solution \mathbf{u} . Indeed, replacing (12) by

$$\begin{cases} \mathbf{K}\mathbf{u} + \mathbf{B}^T \tilde{\boldsymbol{\lambda}} = \tilde{\mathbf{f}} = \mathbf{f} + \mathbf{B}^T \boldsymbol{\mu} \\ \mathbf{B}\mathbf{u} = \mathbf{0} \end{cases} \quad (28)$$

for any $\boldsymbol{\mu}$ will yield the same solution \mathbf{u} , the Lagrange multiplier being then such that $\tilde{\boldsymbol{\lambda}} = \boldsymbol{\lambda} + \boldsymbol{\mu}$. This can also be observed from the fact that (12) and (28) have the same assembled forces since $\mathbf{L}^T \tilde{\mathbf{f}} = \mathbf{L}^T \mathbf{f}$. Mechanically speaking, it means that adding forces on one side of the interface boundary and subtracting it on the other side yields the same solution in terms of displacement although it modifies the internal forces on the interface.

3.1 Force splitting commonly used

From the discussion above, it is clear that one can choose the splitting of interface forces on the interface boundaries¹. Commonly, interface forces are split between the connecting subdomains proportionally to subdomain relative stiffness, in a way consistent with the scaling in the preconditioner. Mathematically speaking, this is equivalent to choosing

$$\tilde{\mathbf{f}} = \text{diag}(\mathbf{K})\mathbf{L} \left(\mathbf{L}^T \text{diag}(\mathbf{K})\mathbf{L} \right)^{-1} \left(\mathbf{L}^T \mathbf{f} \right) \quad (29)$$

where $\left(\mathbf{L}^T \mathbf{f} \right)$ are the applied forces given in an initially assembled problem or the assembled forces corresponding to a different force splitting. $\left(\mathbf{L}^T \text{diag}(\mathbf{K})\mathbf{L} \right)$ is the assembled diagonal stiffness on the interface.

Let us note that for any symmetric positive definite matrix \mathbf{A} , the following

¹ In the same manner, it was noted in [11] that coefficients of additional constraints can also be arbitrarily split on the interface, which led to the construction of efficient preconditioners for problems with multipoint constraints.

relation holds: ²

$$\mathbf{A}\mathbf{L}\left(\mathbf{L}^T\mathbf{A}\mathbf{L}\right)^{-1}\mathbf{L}^T + \mathbf{B}^T\left(\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T\right)^+\mathbf{B}\mathbf{A}^{-1} = \mathbf{I} \quad (30)$$

From (30), one deduces that the force splitting (29) is equivalent to

$$\tilde{\mathbf{f}} = \mathbf{f} - \mathbf{B}^T\left(\mathbf{B}\text{diag}(\mathbf{K})^{-1}\mathbf{B}^T\right)^+\mathbf{B}\text{diag}(\mathbf{K})^{-1}\mathbf{f} \quad (31)$$

This last relation shows that the common splitting technique (29) corresponds to adding a particular set of equilibrated interface forces as described by (28).

3.2 Splitting statically condensed forces

In section 2.4, we indicated that the hybrid decomposed problem can be set in an equivalent form condensed on the interface (21). Following the same splitting procedure for the force as in (29), one would then construct the problem

$$\begin{cases} \mathbf{S}\mathbf{u}_b + \mathbf{B}_b^T\tilde{\boldsymbol{\lambda}} = \tilde{\mathbf{f}}_b^* \\ \mathbf{B}_b\mathbf{u}_b = \mathbf{0} \end{cases} \quad (32)$$

where

$$\tilde{\mathbf{f}}_b^* = \text{diag}(\mathbf{K}_{bb})\mathbf{L}_b\left(\mathbf{L}_b^T\text{diag}(\mathbf{K}_{bb})\mathbf{L}_b\right)^{-1}\left(\mathbf{L}_b^T\mathbf{f}_b^*\right) \quad (33)$$

and where $(\mathbf{L}_b^T\mathbf{f}_b^*)$ are the applied forces statically condensed and assembled on the interface. Following a similar discussion as in the previous section, it is straightforward to show that such a splitting is equivalent to

$$\tilde{\mathbf{f}}_b^* = \mathbf{f}_b^* - \mathbf{B}_b^T\left(\mathbf{B}_b\text{diag}(\mathbf{K}_{bb})^{-1}\mathbf{B}_b^T\right)^+\mathbf{B}_b\text{diag}(\mathbf{K}_{bb})^{-1}\mathbf{f}_b^* \quad (34)$$

indicating again that the splitting corresponds to adding a particular set of equilibrated interface forces.

Let us observe that, in the non-condensed format, (32) is identical to

² This relation expresses the complementarity of the primal and dual scaling and was already suggested in [8]. Its proof can be obtained by noting that $\mathbf{L}^T(30)$ and $\mathbf{B}\mathbf{A}^{-1}(30)$ are trivially satisfied. Owing to the fact that \mathbf{A} is symmetric positive definite and because the image of \mathbf{L} corresponds to the nullspace of \mathbf{B} , the final result is obtained.

$$\begin{aligned}
\mathbf{K}\mathbf{u} + \mathbf{B}^T\boldsymbol{\lambda} &= \begin{bmatrix} \mathbf{f}_i \\ \tilde{\mathbf{f}}_b^* + \mathbf{K}_{bi}\mathbf{K}_{ii}^{-1}\mathbf{f}_i \end{bmatrix} \\
&= \mathbf{f} - \mathbf{B}^T \left(\mathbf{B} \operatorname{diag}(\mathbf{K})^{-1} \mathbf{B}^T \right)^+ \mathbf{B} \operatorname{diag}(\mathbf{K})^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_b^* \end{bmatrix}
\end{aligned} \tag{35}$$

When comparing it with (31), it is clear that splitting the condensed forces is not equivalent to splitting \mathbf{f} as commonly done. Although the displacement field obtained is obviously the same, the Lagrange multipliers searched for during the FETI iterations will be different. The question thus arises: what splitting should be used and how does it affect the FETI iterations on the dual interface problem? Next section provides a theoretical analysis of the interest of the splitting while the section after provides related numerical assessments.

4 An improved initial estimate of the Lagrange multipliers

Let us consider again the hybrid decomposed problem (12). In order to construct an estimate for the interface Lagrange multipliers, let us assume that the internal degrees of freedom \mathbf{u}_i satisfy the local equilibrium while \mathbf{u}_b on the interface boundary have a zero estimate. This is exactly the assumption underlying the initialization of the primal Schur complement iteration schemes. The subdomain equilibrium then writes

$$\mathbf{B}_b^T \boldsymbol{\lambda} \simeq \mathbf{f}_b^* \tag{36}$$

which cannot be exactly satisfied (unless $\mathbf{u}_b = 0$ corresponds to the solution). Hence, let us decompose the statically condensed force \mathbf{f}_b^* into a component satisfying the interface equilibrium (i.e. belonging to the image of \mathbf{B}_b^T) and a remainder:

$$\mathbf{B}_b^T \boldsymbol{\lambda} \simeq \mathbf{f}_b^* = \mathbf{B}_b^T \boldsymbol{\gamma} + \text{remainder} \tag{37}$$

where

$$\boldsymbol{\gamma} = \left(\mathbf{B}_b \mathbf{D} \mathbf{B}_b^T \right)^+ \mathbf{B}_b \mathbf{D} \mathbf{f}_b^* \tag{38}$$

for any non-singular and symmetric matrix \mathbf{D} . The remainder is then orthogonal to $\mathbf{D} \mathbf{B}_b^T$. If we choose as initial estimate

$$\boldsymbol{\lambda}_{00} = \boldsymbol{\gamma} \tag{39}$$

then the initial equilibrium residual $\mathbf{B}_b^T \boldsymbol{\lambda}_{00} - \mathbf{f}_b^* = \text{remainder}$ is minimum in the \mathbf{D} -norm. Setting \mathbf{D} to be $\operatorname{diag}(\mathbf{K}_{bb})^{-1}$

$$\boldsymbol{\lambda}_{00} = \left(\mathbf{B}_b \operatorname{diag}(\mathbf{K}_{bb})^{-1} \mathbf{B}_b^T \right)^+ \mathbf{B}_b \operatorname{diag}(\mathbf{K}_{bb})^{-1} \mathbf{f}_b^* \tag{40}$$

can be easily computed by a scaling of the interface condensed forces. Observing that $\text{diag}(\mathbf{K}_{bb})^{-1}$ is an approximation of \mathbf{S}^+ , $\boldsymbol{\lambda}_0$ minimizes a norm having the meaning of an energy.

We thus propose to start the FETI iterations with an initial estimate obtained by rendering $\boldsymbol{\lambda}_{00}$ admissible, i.e. such that $\mathbf{G}_I^T \boldsymbol{\lambda}_0 = \mathbf{e}$. Using the notations introduced in (17),

$$\boldsymbol{\lambda}_0 = \mathbf{P}(\mathbf{Q})\boldsymbol{\lambda}_{00} + \mathbf{Q}\mathbf{G}_I \left(\mathbf{G}_I^T \mathbf{Q}\mathbf{G}_I \right)^{-1} \mathbf{e} \quad (41)$$

To be consistent with the choice $\mathbf{D} = \text{diag}(\mathbf{K}_{bb})^{-1}$, one should take \mathbf{Q} as in (26) or (27).

Remarks

- If $\mathbf{D} = \mathbf{S}^+$ in (38), $\boldsymbol{\lambda}_{00} = \mathbf{F}_I^+ \mathbf{d}$. Furthermore, one would have $\mathbf{Q} = \mathbf{F}_I^+$ and

$$\begin{aligned} \boldsymbol{\lambda}_0 &= \mathbf{P}(\mathbf{Q})\mathbf{F}_I^+ \mathbf{d} + \mathbf{F}_I^+ \mathbf{G}_I \left(\mathbf{G}_I^T \mathbf{F}_I^+ \mathbf{G}_I \right)^{-1} \mathbf{e} \\ &= \mathbf{F}_I^+ (\mathbf{d} - \mathbf{G}_I \boldsymbol{\alpha}) \\ &\text{with } \boldsymbol{\alpha} = \left(\mathbf{G}_I^T \mathbf{F}_I^+ \mathbf{G}_I \right)^{-1} (\mathbf{G}_I^T \mathbf{F}_I^+ \mathbf{d} - \mathbf{e}) \end{aligned} \quad (42)$$

Hence the new choice of initial estimate would yield the exact solution.

- If we choose to split the interface forces such as described in (33) (or its equivalent form 34), (40) yields $\boldsymbol{\lambda}_{00} = \mathbf{0}$ so that the initial iterate (41) would correspond to the standard FETI starting procedure. Hence *it is equivalent to apply standard FETI iterations when splitting condensed forces on the interface as in (33) or to use any decomposed force vector \mathbf{f} together with the starting procedure (41)*.
- The construction of $\boldsymbol{\lambda}_{00}$ in (40) has exactly the same mechanical interpretation as the scaling of interface forces in the preconditioner (see [11]). The cost incurred by (33) or (41), corresponds to a preconditioning step, thus less than one half of a FETI iteration. Obviously, if the lumped preconditioner is applied, the standard splitting (29) should be considered.

5 Numerical assessment

In order to assess the performance of the new estimate for the Lagrange multipliers, we consider the solution to various problems by FETI and BDD. In all

cases, convergence is monitored through the evaluation of the global residual:

$$\frac{\|\mathbf{K}_g \mathbf{u}_g - \mathbf{f}_g\|}{\|\mathbf{f}_g\|} \leq \varepsilon \quad (43)$$

where \mathbf{K}_g , \mathbf{u}_g and \mathbf{f}_g are global assembled stiffness matrix, displacement field and forces (see (13)). All results are obtained with the Dirichlet preconditioner equipped with the super-lumped scaling, i.e. $\mathbf{A} = \text{diag}(\mathbf{K}_{bb})^{-1}$. The stopping criterion ε is set to 10^{-6} . The FETI method is tested for its different projectors: we note $\mathbf{P}(\tilde{\mathbf{F}}_{I_D}^{-1})$ the Dirichlet projector, $\mathbf{P}(\mathbf{W})$ the superlumped projector corresponding to (27) and $\mathbf{P}(\mathbf{I})$ the identity projector.

5.0.0.1 Cube with checkerboard heterogeneity: Let us first assess the strategies described above on the problem described in the introduction (see fig. 1). Figure 4 presents the convergence of the primal residual through the conjugate gradient iterations for the BDD and for the FETI approaches with standard force splitting. The primal approach clearly exhibit better results. Also, due to the strong heterogeneity in the structure and along the interfaces, the projector $\mathbf{P}(\mathbf{I})$ yields very poor convergence and is thus not suitable. It is observed that, although the Dirichlet projector yields a better convergence rate, the superlumped projector $\mathbf{P}(\mathbf{W})$ has a significantly lower initial residual and hence reaches convergence faster. The initialization associated to $\mathbf{P}(\mathbf{W})$ leads to a starting residual 10^3 times smaller, however its behaviour during the solution process is less regular.

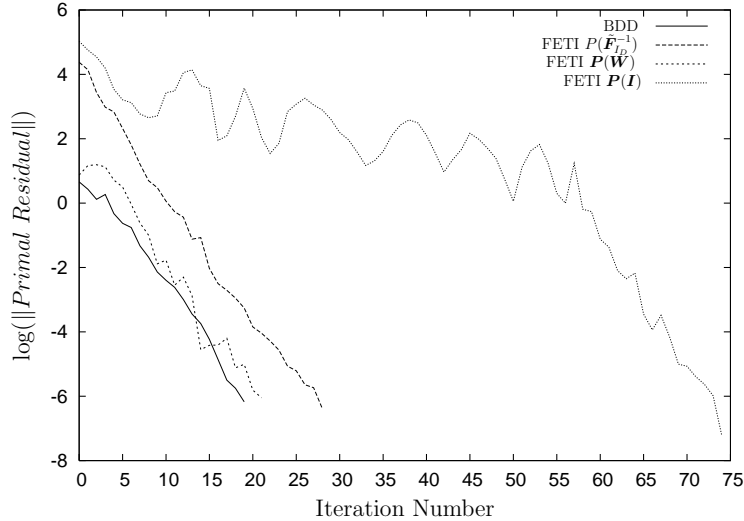


Fig. 4. Cube with checkerboard heterogeneity (fig. 1): convergence of the primal residual for FETI with standard force splitting

Figure 5 presents the convergence history for the BDD method and for the FETI method equipped with the Dirichlet projector and with the new initialization. For comparison, the convergence obtained with the standard splitting

is shown once more. Note that the new initialization not only provides a better initial estimate (lower initial residual) but also leads to similar convergence rates as for the standard splitting. Therefore the convergence history of FETI becomes very similar to the convergence of the primal BDD approach. The total number of iterations is about the same as for the primal BDD, although the convergence of FETI is less monotonic.

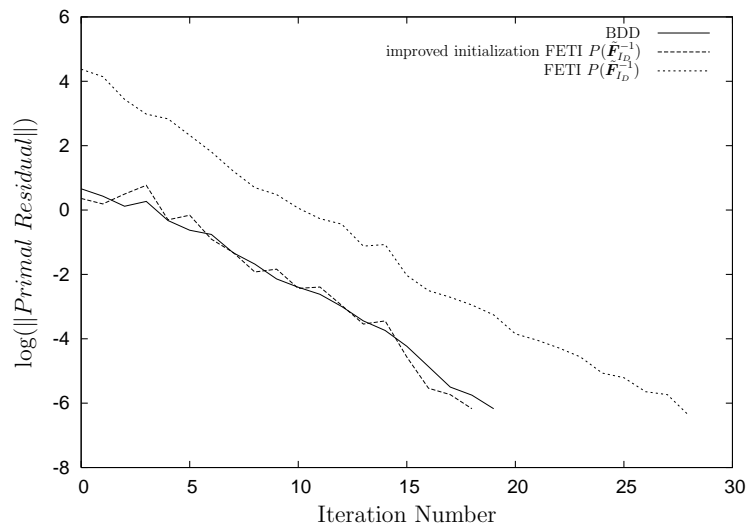


Fig. 5. Cube with checkerboard heterogeneity (fig. 1): convergence of the primal residual for FETI with new initialization

Approach		Number of iterations	Initial Residual (Log)
BDD		19	0.659
FETI Dirichlet $P(\tilde{F}_{I_D}^{-1})$	No Splitting	28	4.428
	Classical Splitting	28	4.377
	New Initialization	18	0.359
FETI Dirichlet $P(W)$	No Splitting	21	0.873
	Classical Splitting	21	0.872
	New Initialization	20	0.868
FETI Dirichlet $P(I)$	No Splitting	74	5.029
	Classical Splitting	74	5.026
	New Initialization	73	5.016

Table 1

Performance results for the cube of fig. 1

Table 1 summarizes the performance results of the various available strategies. To investigate the efficiency of the new initialization (or force splitting)

procedure, we compare the number of iterations to achieve convergence and the norm of the initial primal residual. The new initialization yields significant improvements for the Dirichlet projector (35% less iterations), but only slightly affects the convergence when other projectors are used.

5.0.0.2 Cube and slanted cube with heterogeneous layers: We now assess the new initialization strategy on two other configurations of the cube in order to evaluate the influence of the geometry and of the repartition of heterogeneities. The structures depicted in Figure 6 and 7 are similar to fig. 1 but with different material distribution. Also, for the problem described in Figure 7, the cube has been slanted by 60 degrees. In table 2 we report the number of iterations when using the BDD solver and when applying FETI with the standard and the new initialization.

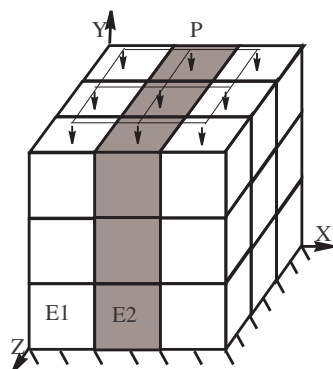


Fig. 6. Cube with heterogeneous layers

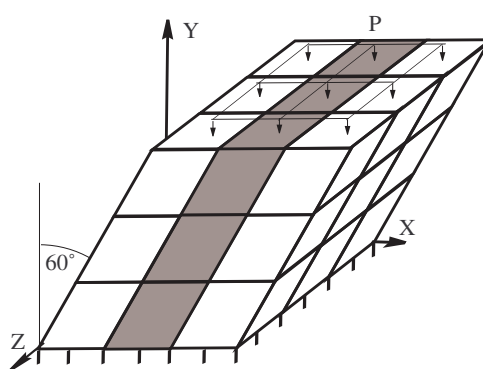


Fig. 7. Slanted cube with heterogeneous layers

For the straight cube of fig. 6, the use of the novel initialization leads to only small improvements. Observing that the FETI solver with Dirichlet projector and standard splitting converges nearly as fast as the BDD, it is clear that the new splitting strategy can not have a significant effect.

For the slanted cube, due to the geometric distortion of the substructures and of the mesh, the BDD convergence is better than when the FETI with the standard splitting is applied. When the new initialization strategy is used, the convergence history of FETI is again very similar to the convergence of the primal BDD.

5.0.0.3 Nonlinear flexion of a composite beam As a last example, we analyze a non-linear problem solved through a sequence of linearized systems. The structure (fig. 8) is a slender beam of aspect ratio 9 with square cross-section. It is made of longitudinal strips of metal and rubber. The loading corresponds to an imposed pressure on one side of the beam. Due to the presence of elastomer parts, the structure undergoes large deformations. We chose

Approach		Cube, fig. 6 Num. of it.	Slanted cube, fig. 7 Num. of it.
BDD		19	73
FETI Dirichlet $P(\tilde{F}_{I_D}^{-1})$	No Splitting	20	85
	Classical Splitting	21	85
	New Initialization	19	73
FETI Dirichlet $P(W)$	No Splitting	22	88
	Classical Splitting	22	88
	New Initialization	22	85
FETI Dirichlet $P(I)$	No Splitting	92	153
	Classical Splitting	92	154
	New Initialization	90	159

Table 2

Performance results on problem fig. 6 and 7

a Kirchoff Saint-Venant model for the metal, the characteristic coefficients of which are its Young modulus $E = 20000$ MPa and Poisson's coefficient $\nu = 0.3$. A NeoHookian model is assumed for the rubber characterized by a shear modulus $G = 2.0$ MPa and a compressibility modulus $K = 2000$ MPa. The structure is decomposed into 27 monomaterial parallelepipedic subdomains and each substructure is meshed in $2 \times 2 \times 18$ cubic elements. In order to handle the quasi-incompressibility of the elastomer, a mixed finite element formulation is considered where the pressure field and displacements are discretized independently. We choose a $Q_2 - P_1$ hexaedral element with 27 displacement nodes and 4 internal pressure nodes. Details on this formulation and on the practicalities for applying it properly in simulation can be found in [17].

The complete model has 55300 degrees of freedom of which 16400 belong to the interface. Due to the behaviour of the elastomer, the problem is highly nonlinear. Newton-Raphson iterations are performed where the tangent matrix is updated at every step. The pressure loading is 5 bars. The stopping criterion is set to 10^{-3} for the relative primal residual of FETI when solving the linearized systems. For the Newton-Raphson iterations, the tolerance for the relative residual of the nonlinear equations is set to 10^{-4} , so that 5 Newton-Raphson iterations and thus 5 linear solves must be performed.

Figure 9 indicates the number of iterations required when solving the linearized systems by the classical FETI method, by FETI with improved initialization and by the primal BDD. The Dirichlet and diagonal projectors are applied,

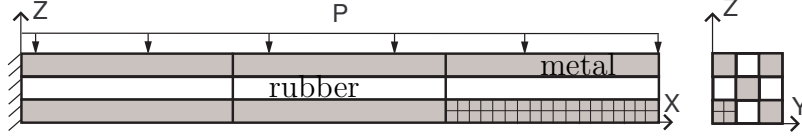


Fig. 8. $3 \times 3 \times 3$ substructures composite beam

and the Dirichlet preconditioner with stiffness scaling is used. The significant increase of the number of iterations between the first and the following linearized systems can be explained by the loss of positivity of the tangent matrix which is mostly due to incompressibility. Indeed, the conjugate gradient algorithm (with full re-orthogonalization) applied on the interface problem in the FETI and in the BDD methods remains applicable for non positive matrices but its convergence is significantly slowed down [18].³ As observed from figure 9, the new initialization enables FETI to achieve performance results which are very similar to the BDD method. Comparing the total number of iterations, the classical FETI approach requires 10% more iterations than BDD while the FETI method with new initialization requires slightly less iterations than the BDD. From figure 9 we also observe that the new initialization technique improves the performance of FETI both for the Dirichlet and the diagonal projector. For this particular structure with regular geometry, the cost effective diagonal projector yields a convergence rate very similar to the convergence rate obtained with the more computationally intensive Dirichlet projector, except for the very first linearized system solve.

Our numerical experiments indicate that for a large class of nonlinear problems such as the one depicted here, the new initialization leads to a small but non-negligible gain in terms of number of iterations and CPU time. Another important beneficial effect of the new starting procedure for FETI comes from the fact that, since the initial residual of the iterations on the interface problem are several orders of magnitude lower with the new initialization, stagnation of the residual of the FETI iterations which often happens when dealing with higher nonlinearity (higher loading) is significantly delayed, so that in practice restarting of the iteration can be avoided.

³ Often, non-positivity is due to the mutation of former null-modes (rotations) to negative modes, it can be handled by the introduction of these modes as constraints in Krylov-augmented algorithms [19]. In our case, non-positivity is mostly due to the behaviour of the rubber and the strategy described above is non-relevant. An efficient strategy based on the approximation of negative modes can be found in [20].

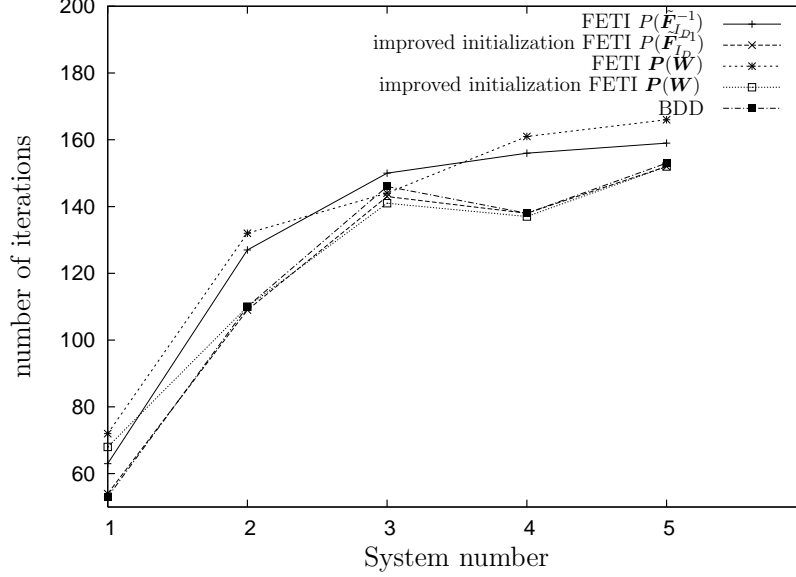


Fig. 9. Performance results for the flexion of the composite beam

6 Conclusions

For some particular structural problems such as those exhibiting strong heterogeneities and geometric distortion, the Finite Element Tearing and Interconnecting (FETI) solver can yield poor convergence compared to the conceptually similar Balanced Domain Decomposition (BDD) solver. In those cases, the bad performance of FETI can be traced back to high initial residual in the iterations on the interface problem. The initial residual is strongly related to the way the applied forces are split on the subdomain interface boundaries.

In this paper, we propose a novel strategy to split the applied forces between the subdomains. We propose to split the statically condensed interface force according to interface diagonal stiffness. This leads to building a more efficient initial estimate for the interface connecting forces. The new initialization for the FETI iterations mainly involves computing statically condensed forces on the interface and can thus be performed at a computational cost equivalent to less than half the cost of a full FETI iteration.

The numerical examples described in this paper indicate that for the problems where the primal BDD method outperforms FETI due to unexpected high initial residual, the new initialization strategy builds a better starting estimate of the interface forces and, in turn, to an initial residual similar to the BDD residual. The FETI method then converges in a manner very similar to the BDD method.

The novel starting strategy never deteriorates the FETI convergence and leads to significant improvements in some pathological cases. Therefore we suggest

to use the presented initialization as default in FETI solvers.

With the proposed initialization for FETI, the FETI method and the BDD solver lead to similar convergence and computational costs for complex problem where the Dirichlet projector is required. For problems where the simplified FETI preconditioners and projectors can be used without significantly deteriorating the convergence of the interface iterations, FETI is often found to be more efficient in terms of overall computing cost.

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On the initial estimate of interface forces in FETI methods

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Abstract

The Balanced Domain Decomposition (BDD) method and the Finite Element Tearing and Interconnecting (FETI) method are two commonly used non-overlapping domain decomposition methods. Due to strong theoretical and numerical similarities, these two methods are generally considered as being equivalently efficient. However, for some particular cases, such as for structures with strong heterogeneities, FETI requires a large number of iterations to compute the solution compared to BDD. In this paper, the origin of the bad efficiency of FETI in these particular cases is traced back to poor initial estimates of the interface stresses. To improve the estimation of interface forces a novel strategy for splitting interface forces between neighboring substructures is proposed. The additional computational cost incurred is not significant. This yields a new initialization for the FETI method and restores numerical efficiency which makes FETI comparable to BDD even for problems where FETI was performing poorly. Various simple test problems are presented to discuss the efficiency of the proposed strategy and to illustrate the so-obtained numerical equivalence between the BDD and FETI solvers.

Key words: domain decomposition, iterative solver, FETI, Schur complement, force splitting

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1 Introduction

Domain Decomposition methods provide a natural framework to solve engineering problems decomposed into subparts. Such problems can arise for instance because each subdomain is discretized independently, or because the subdomains represent different physical domains. Decomposed domains can also be created from an initial single domain problem in order to make efficient use of parallel computing hardware, i.e. in order to distribute the computing work on several processors.

Among the domain decomposition methods applied in engineering mechanics to solve elliptic linear problems, two similar methods have emerged in the last decade as efficient parallel computing methods: the primal and the dual Schur complement methods. More specifically, two procedures have been shown to ensure scalability and robustness: the Balanced Domain Decomposition (BDD) method [?, ?] and the Finite Element Tearing and Interconnecting (FETI) method [?, ?]. The BDD method is a primal procedure where preconditioned conjugate gradient iterations are applied to find the interface displacements that satisfy the interface equilibrium. In the FETI approach, it is the interface forces that are searched for iteratively so as to satisfy the interface displacement compatibility. Therefore FETI is sometimes referred to as a dual method.

The primal and dual Schur complement methods are based on very similar concepts (see e.g. [?] for a mechanical description). Mathematically, it has been shown that the preconditioned interface operators for both the BDD and FETI methods have a condition number bounded by [?, ?]

$$\kappa = O\left(1 + \log\frac{H}{h}\right)^2 \quad (1)$$

where H and h represent the subdomain and the mesh size respectively. Hence it is often accepted in the Domain Decomposition community that using one method instead of the other is a matter of taste and implementation preferences. However, when looking closely at the details and variants of both methods, it becomes rapidly clear that showing the exact equivalence between the primal BDD and the dual FETI is not easy, if at all possible [?].

Let us then consider the simple example depicted in Figure 1 of a highly heterogeneous elastic cube ($\frac{E_1}{E_2} = 10^5$) subdivided into $3 \times 3 \times 3$ subdomains. A uniform pressure is applied on the face opposite to the clamped side. The structure is discretized using Q_2 hexahedral finite elements with 27 nodes. The model contains 21000 degrees of freedom, of which 6000 belong to subdomain interfaces. In Figure 2 the convergence curves of the global equilibrium residual corresponding to iterations of the BDD and the FETI methods are

plotted. Both methods are equipped with what literature refers to as the best preconditioners and coarse grids (see section 2). Although the asymptotic convergence of both procedures is similar, it is observed that the convergence of the FETI method is less monotonic and that its initial residual is significantly higher.

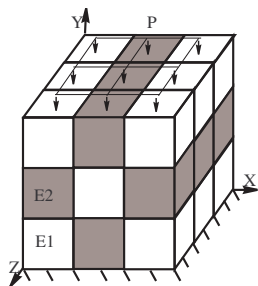


Fig. 1. Decomposed heterogeneous cube

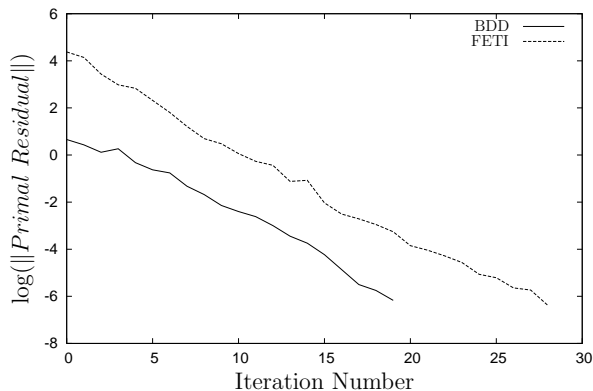


Fig. 2. Convergence of BDD & FETI

These results seem to indicate that some specific details such as the choice of the initial estimates result in possibly significant differences between the primal and dual algorithms.

In this paper we revisit the way the interface forces are estimated initially in the FETI method in an attempt to obtain a convergence comparable to that of the BDD.

In the next section, we shortly recall the concepts underlying the FETI solver. In section 3 we explain that the forces applied on the interface can be split in different ways. Although the final result is independent of that splitting, it affects the actual FETI iteration history. We then present an efficient way to define such a splitting and show how it is related to the construction of the initial iterate of FETI in section 4. Numerical examples are reported in section 5 to illustrate the effectiveness of the new initialization strategy. Finally, we present some conclusions.

2 FETI basics

2.1 The decomposed problem

Let us consider a domain Ω subdivided into N_s non-overlapping subdomains $\Omega^{(s)}$ and assume that we are solving a linear (or linearized) static equilibrium

problem on the domain. The discretized subdomain equilibrium is expressed by

$$\mathbf{K}^{(s)}\mathbf{u}^{(s)} = \mathbf{f}^{(s)} + \mathbf{g}^{(s)} \quad s = 1, \dots, N_s \quad (2)$$

where $\mathbf{K}^{(s)}$, $\mathbf{u}^{(s)}$ and $\mathbf{f}^{(s)}$ are the subdomain stiffness matrices, displacements and applied forces respectively. $\mathbf{g}^{(s)}$ are the connecting forces on the interface between subdomains (thus zero on the internal degrees of freedom). For the sake of simplicity, we assume in the following that the meshes are matching (conforming) on the interface.

The interface forces satisfy an interface equilibrium equation expressing that when assembled on the interface, the resultant is null (action-reaction):

$$\sum_{s=1}^{N_s} \mathbf{L}^{(s)T} \mathbf{g}^{(s)} = \mathbf{0} \quad (3)$$

where $\mathbf{L}^{(s)}$ is a Boolean assembly matrix. The interface connecting forces are such that the interface degrees of freedom are compatible, namely

$$\sum_{s=1}^{N_s} \mathbf{B}^{(s)} \mathbf{u}^{(s)} = \mathbf{0} \quad (4)$$

This relation expresses that for any pair $(u^{(s)}, u^{(r)})$ of degrees of freedom matching and the interface, $u^{(s)} - u^{(r)} = 0$. $\mathbf{B}^{(s)}$ are thus signed Boolean matrices expressing the compatibility constraints on the interface.

The equilibrium problem of domain Ω is fully described by the local equilibrium (2) and by the interface constraints (3, 4). In block diagonal notations, it can be summarized as

$$\begin{cases} \mathbf{K}\mathbf{u} = \mathbf{f} + \mathbf{g} \\ \mathbf{L}^T \mathbf{g} = \mathbf{0} \\ \mathbf{B}\mathbf{u} = \mathbf{0} \end{cases} \quad (5)$$

where \mathbf{K} is the block diagonal matrix of the local operators $\mathbf{K}^{(s)}$ and where

$$\begin{aligned} \mathbf{u} &= \begin{bmatrix} \mathbf{u}^{(1)} \\ \vdots \\ \mathbf{u}^{(N_s)} \end{bmatrix} & \mathbf{f} &= \begin{bmatrix} \mathbf{f}^{(1)} \\ \vdots \\ \mathbf{f}^{(N_s)} \end{bmatrix} & \mathbf{g} &= \begin{bmatrix} \mathbf{g}^{(1)} \\ \vdots \\ \mathbf{g}^{(N_s)} \end{bmatrix} \\ \mathbf{L}^T &= \begin{bmatrix} \mathbf{L}^{(1)T} & \dots & \mathbf{L}^{(N_s)T} \end{bmatrix} \\ \mathbf{B} &= \begin{bmatrix} \mathbf{B}^{(1)} & \dots & \mathbf{B}^{(N_s)} \end{bmatrix} \end{aligned} \quad (6)$$

Note that in this description, one set of interface displacements and one set of interface forces are defined per subdomain.

2.2 Solvers for decomposed problems

Solving (5) can be done in several ways:

- Considering (5) as a constrained equilibrium problem in terms of \mathbf{u} and \mathbf{g} leads to the three-field formulation of decomposed domains (see e.g. [?, ?]).
- One can choose to work with a displacement set \mathbf{u} that satisfies a priori the interface compatibility (4). For that purpose we define a global set \mathbf{u}_g of degrees of freedom unique on the interface such that

$$\mathbf{u}^{(s)} = \mathbf{L}^{(s)} \mathbf{u}_g \quad \text{or} \quad \mathbf{u} = \mathbf{L} \mathbf{u}_g \quad (7)$$

where $\mathbf{L}^{(s)}$ is the same assembly Boolean matrix as in (3) that extracts subdomain degrees of freedom from the global set. Stating that $\mathbf{u}^{(s)}$ are obtained from a unique set is obviously equivalent to stating the interface compatibility (4) and (7) thus implies

$$\mathbf{B} \mathbf{u} = \mathbf{B} \mathbf{L} \mathbf{u}_g = \mathbf{0} \quad (8)$$

for any global displacement \mathbf{u}_g . On the other hand, all compatible displacements can be written as in (7). Hence

$$\mathbf{L} = \text{null}(\mathbf{B}) \quad (9)$$

In order to illustrate these concepts, we reproduce the example of [?] in Figure 3. Introducing (7) in (5) yields

$$\begin{cases} \mathbf{K} \mathbf{L} \mathbf{u}_g = \mathbf{f} + \mathbf{g} \\ \mathbf{L}^T \mathbf{g} = \mathbf{0} \end{cases} \quad (10)$$

This set of equations is at the basis of the primal iterative solution techniques such as the Primal Schur Complement or the BDD methods [?, ?]: iteration schemes are applied to find the displacements \mathbf{u}_g until the interface equilibrium $\mathbf{L}^T \mathbf{g} = \mathbf{L}^T (\mathbf{K} \mathbf{L} \mathbf{u}_g - \mathbf{f}) = \mathbf{0}$ is satisfied.

- One can choose in (5) a set of interface forces satisfying a priori the interface equilibrium $\mathbf{L}^T \mathbf{g} = \mathbf{0}$ while keeping redundant interface degrees of freedom in \mathbf{u} . According to (9), such interface forces have the generic expression

$$\mathbf{g}^{(s)} = -\mathbf{B}^{(s)T} \boldsymbol{\lambda} \quad \text{or} \quad \mathbf{g} = -\mathbf{B}^T \boldsymbol{\lambda} \quad (11)$$

$\boldsymbol{\lambda}$ are interface forces that act in opposite directions between any pair of matching degrees of freedom on the interface and are therefore in equilibrium

$$\underbrace{\begin{bmatrix} \overbrace{-1 \ 0}^{B^{(1)}} & \overbrace{1 \ 0}^{B^{(2)}} & \overbrace{0 \ 0}^{B^{(3)}} \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}}_B \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}}_L \begin{bmatrix} u_{g1} \\ u_{g2} \end{bmatrix}$$

Fig. 3. Lagrange multipliers and interface compatibility

(see Figure 3). Problem (5) becomes

$$\begin{cases} \mathbf{K}\mathbf{u} + \mathbf{B}^T\boldsymbol{\lambda} = \mathbf{f} \\ \mathbf{B}\mathbf{u} = \mathbf{0} \end{cases} \quad (12)$$

Clearly, $\boldsymbol{\lambda}$ are the Lagrange multipliers associated to the interface compatibility constraints. This form of the decomposed problem is the basis for the dual procedures such as FETI: iterative algorithms are applied to compute the interface forces $\boldsymbol{\lambda}$ such that the displacements resulting from the subdomain equilibrium are compatible on the interface.

- If one chooses interface displacements that are unique on part of the interface while, on the remainder of the interface, equilibrated connecting forces are defined, one obtains hybrid primal/dual approaches such as the FETI-DP procedure [?].
- If on the entire interface we use displacements and forces that satisfy a linear

combination of the interface equilibrium and compatibility (e.g. Robin type of boundary conditions), one obtains formulations typically used in wave propagation analysis such as described for instance by Helmholtz equations [?].

- Finally if both the interface equilibrium and compatibility are enforced a priori, one obtains the fully assembled form

$$\mathbf{L}^T \mathbf{K} \mathbf{L} \mathbf{u}_g = \mathbf{L}^T \mathbf{f} \quad (13)$$

2.3 FETI: the dual iterative solver

In the FETI method [?], the decomposed problem (12) is expressed in terms of interface forces $\boldsymbol{\lambda}$: using the subdomain equilibrium equations to eliminate $\mathbf{u}^{(s)}$,

$$\mathbf{u}^{(s)} = \mathbf{K}^{(s)+} \left(\mathbf{f}^{(s)} - \mathbf{B}^{(s)T} \boldsymbol{\lambda} \right) - \mathbf{R}^{(s)} \boldsymbol{\alpha}^{(s)} \quad (14)$$

where $\mathbf{K}^{(s)+}$ is the inverse of $\mathbf{K}^{(s)}$ or a generalized inverse if subdomain $\Omega^{(s)}$ is floating when disconnected from its neighbors. In the latter case, $\mathbf{R}^{(s)}$ are the associated rigid body modes, their amplitudes $\boldsymbol{\alpha}^{(s)}$ being determined such that the interface forces are in equilibrium with the applied forces $\mathbf{f}^{(s)}$, i.e. such that the subdomain equilibrium is well-posed:

$$\mathbf{R}^{(s)T} \left(\mathbf{f}^{(s)} - \mathbf{B}^{(s)T} \boldsymbol{\lambda} \right) = \mathbf{0} \quad (15)$$

Substituting (14) into the interface compatibility condition, and taking account of (15), one obtains the dual interface problem

$$\begin{bmatrix} \mathbf{F}_I & \mathbf{G}_I \\ \mathbf{G}_I^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\alpha} \end{bmatrix} = \begin{bmatrix} \mathbf{d} \\ \mathbf{e} \end{bmatrix} \quad (16)$$

where

$$\begin{aligned}
\mathbf{F}_I &= \sum_{s=1}^{N_s} \mathbf{B}^{(s)} \mathbf{K}^{(s)+} \mathbf{B}^{(s)T} = \mathbf{B} \mathbf{K}^+ \mathbf{B}^T \\
\mathbf{d} &= \sum_{s=1}^{N_s} \mathbf{B}^{(s)} \mathbf{K}^{(s)+} \mathbf{f}^{(s)} = \mathbf{B} \mathbf{K}^+ \mathbf{f} \\
\mathbf{G}_I &= \begin{bmatrix} \mathbf{B}^{(1)} \mathbf{R}^{(1)} & \dots & \mathbf{B}^{(N_s)} \mathbf{R}^{(N_s)} \end{bmatrix} = \mathbf{B} \mathbf{R} \\
\boldsymbol{\alpha} &= \begin{bmatrix} \boldsymbol{\alpha}^{(1)} \\ \vdots \\ \boldsymbol{\alpha}^{(N_s)} \end{bmatrix} \quad \text{and} \quad \mathbf{e} = \begin{bmatrix} \mathbf{R}^{(1)T} \mathbf{f}^{(1)} \\ \vdots \\ \mathbf{R}^{(N_s)T} \mathbf{f}^{(N_s)} \end{bmatrix} = \mathbf{R}^T \mathbf{f}
\end{aligned}$$

Let us define the projection operator \mathbf{P} such that $\mathbf{G}_I^T \mathbf{P} = \mathbf{0}$:

$$\mathbf{P}(\mathbf{Q}) = \mathbf{I} - \mathbf{Q} \mathbf{G}_I \left(\mathbf{G}_I^T \mathbf{Q} \mathbf{G}_I \right)^{-1} \mathbf{G}_I^T \quad (17)$$

The choice of the operator \mathbf{Q} is discussed later. Introducing the splitting

$$\boldsymbol{\lambda} = \mathbf{P}(\mathbf{Q}) \bar{\boldsymbol{\lambda}} + \boldsymbol{\lambda}_0 \quad (18)$$

$$\boldsymbol{\lambda}_0 = \mathbf{Q} \mathbf{G}_I \left(\mathbf{G}_I^T \mathbf{Q} \mathbf{G}_I \right)^{-1} \mathbf{e} \quad (19)$$

the dual interface problem (16) is equivalent to

$$\mathbf{P}(\mathbf{Q})^T \mathbf{F}_I \mathbf{P}(\mathbf{Q}) \bar{\boldsymbol{\lambda}} = \mathbf{P}(\mathbf{Q})^T (\mathbf{d} - \mathbf{F}_I \boldsymbol{\lambda}_0) \quad (20)$$

The FETI method consists in preconditioned conjugate gradient iterations on the dual interface problem (20) [?]. Applying \mathbf{F}_I to an iterate is a naturally parallel operation. The projection steps however require solving a coarse global problem. The preconditioning operators and the possible choices for \mathbf{Q} are summarized next.

2.4 FETI preconditioners and coarse grid space

The forces $\mathbf{B}^T \boldsymbol{\lambda}$ exist only on the interface degrees of freedom. Using a subscript b and i for interface boundary and internal degrees of freedom respectively, one can re-write the decomposed problem (12) by condensing out \mathbf{u}_i :

$$\begin{cases} \mathbf{S} \mathbf{u}_b + \mathbf{B}_b^T \boldsymbol{\lambda} = \mathbf{f}_b^* \\ \mathbf{B}_b \mathbf{u}_b = \mathbf{0} \end{cases} \quad (21)$$

where \mathbf{B}_b is a submatrix of \mathbf{B} associated to boundary displacements only and

$$\mathbf{S} = \begin{bmatrix} & & \ddots & & \\ & & & & \\ & \mathbf{K}_{bb}^{(s)} - \mathbf{K}_{bi}^{(s)} \mathbf{K}_{ii}^{(s)-1} \mathbf{K}_{ib}^{(s)} & & & \\ & & & & \\ & & & \ddots & \end{bmatrix} \quad (22)$$

$$\mathbf{f}_b^* = \begin{bmatrix} \vdots \\ \mathbf{f}_b^{(s)} - \mathbf{K}_{bi}^{(s)} \mathbf{K}_{ii}^{(s)-1} \mathbf{f}_i^{(s)} \\ \vdots \end{bmatrix} \quad (23)$$

\mathbf{S} is the block diagonal matrix of the local operators statically condensed on the interface (also known as Schur complements). The dual interface problem (16) or (20) can thus be written in equivalent forms by replacing \mathbf{K} , \mathbf{f} and \mathbf{B} by \mathbf{S} , \mathbf{f}_b^* and \mathbf{B}_b respectively.

The preconditioner in FETI for which the optimal conditioning number (1) holds is the Dirichlet preconditioner:

$$\tilde{\mathbf{F}}_{I_D}^{-1} = \tilde{\mathbf{B}}_b \mathbf{S} \tilde{\mathbf{B}}_b^T \quad (24)$$

The operator $\tilde{\mathbf{B}}_b$ is similar to \mathbf{B}_b but includes a scaling with respect to interface multiplicity or relative interface stiffness [?]. The scaling is implemented as a simple pre- and post-processing in the preconditioning step. Mathematically, its general expression is [?, ?]

$$\tilde{\mathbf{B}}_b = \left(\mathbf{B}_b \mathbf{A} \mathbf{B}_b^T \right)^+ \mathbf{B}_b \mathbf{A} \quad (25)$$

where the $^+$ superscript denotes an inverse or a pseudo-inverse in case redundant compatibility constraints are present. It can be shown that the multiplicity scaling procedure corresponds to $\mathbf{A} = \mathbf{I}$, whereas the stiffness scaling (also known as super-lumped scaling) corresponds to $\mathbf{A} = \text{diag}(\mathbf{K}_{bb})^{-1}$.

The operator \mathbf{Q} in the coarse grid projector (17) defines the interface forces $\mathbf{Q}\mathbf{G}_I$ associated to rigid mode displacements of subdomains. It can be chosen as $\mathbf{Q} = \mathbf{I}$, but for many engineering problems it should be taken as the preconditioner, namely

$$\mathbf{Q} = \tilde{\mathbf{F}}_{I_D}^{-1} \quad (26)$$

or at least as a degenerated form of the preconditioner such as

$$\mathbf{Q} = \left(\mathbf{B}_b \text{diag}(\mathbf{K}_{bb})^{-1} \mathbf{B}_b^T \right)^+ \quad (27)$$

Further discussion on preconditioning, scaling and on the choice of \mathbf{Q} can be

found in [?, ?, ?, ?].

3 Splitting forces on an interface

The definition of the decomposed hybrid problem (12) is not unique: any interface force satisfying the interface equilibrium can be added to the system without changing the final solution \mathbf{u} . Indeed, replacing (12) by

$$\begin{cases} \mathbf{K}\mathbf{u} + \mathbf{B}^T \tilde{\boldsymbol{\lambda}} = \tilde{\mathbf{f}} = \mathbf{f} + \mathbf{B}^T \boldsymbol{\mu} \\ \mathbf{B}\mathbf{u} = \mathbf{0} \end{cases} \quad (28)$$

for any $\boldsymbol{\mu}$ will yield the same solution \mathbf{u} , the Lagrange multiplier being then such that $\tilde{\boldsymbol{\lambda}} = \boldsymbol{\lambda} + \boldsymbol{\mu}$. This can also be observed from the fact that (12) and (28) have the same assembled forces since $\mathbf{L}^T \tilde{\mathbf{f}} = \mathbf{L}^T \mathbf{f}$. Mechanically speaking, it means that adding forces on one side of the interface boundary and subtracting it on the other side yields the same solution in terms of displacement although it modifies the internal forces on the interface.

3.1 Force splitting commonly used

From the discussion above, it is clear that one can choose the splitting of interface forces on the interface boundaries¹. Commonly, interface forces are split between the connecting subdomains proportionally to subdomain relative stiffness, in a way consistent with the scaling in the preconditioner. Mathematically speaking, this is equivalent to choosing

$$\tilde{\mathbf{f}} = \text{diag}(\mathbf{K})\mathbf{L} \left(\mathbf{L}^T \text{diag}(\mathbf{K})\mathbf{L} \right)^{-1} \left(\mathbf{L}^T \mathbf{f} \right) \quad (29)$$

where $\left(\mathbf{L}^T \mathbf{f} \right)$ are the applied forces given in an initially assembled problem or the assembled forces corresponding to a different force splitting. $\left(\mathbf{L}^T \text{diag}(\mathbf{K})\mathbf{L} \right)$ is the assembled diagonal stiffness on the interface.

Let us note that for any symmetric positive definite matrix \mathbf{A} , the following

¹ In the same manner, it was noted in [?] that coefficients of additional constraints can also be arbitrarily split on the interface, which led to the construction of efficient preconditioners for problems with multipoint constraints.

relation holds: ²

$$\mathbf{A}\mathbf{L}\left(\mathbf{L}^T\mathbf{A}\mathbf{L}\right)^{-1}\mathbf{L}^T + \mathbf{B}^T\left(\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T\right)^+\mathbf{B}\mathbf{A}^{-1} = \mathbf{I} \quad (30)$$

From (30), one deduces that the force splitting (29) is equivalent to

$$\tilde{\mathbf{f}} = \mathbf{f} - \mathbf{B}^T\left(\mathbf{B}\text{diag}(\mathbf{K})^{-1}\mathbf{B}^T\right)^+\mathbf{B}\text{diag}(\mathbf{K})^{-1}\mathbf{f} \quad (31)$$

This last relation shows that the common splitting technique (29) corresponds to adding a particular set of equilibrated interface forces as described by (28).

3.2 Splitting statically condensed forces

In section 2.4, we indicated that the hybrid decomposed problem can be set in an equivalent form condensed on the interface (21). Following the same splitting procedure for the force as in (29), one would then construct the problem

$$\begin{cases} \mathbf{S}\mathbf{u}_b + \mathbf{B}_b^T\tilde{\boldsymbol{\lambda}} = \tilde{\mathbf{f}}_b^* \\ \mathbf{B}_b\mathbf{u}_b = \mathbf{0} \end{cases} \quad (32)$$

where

$$\tilde{\mathbf{f}}_b^* = \text{diag}(\mathbf{K}_{bb})\mathbf{L}_b\left(\mathbf{L}_b^T\text{diag}(\mathbf{K}_{bb})\mathbf{L}_b\right)^{-1}\left(\mathbf{L}_b^T\mathbf{f}_b^*\right) \quad (33)$$

and where $(\mathbf{L}_b^T\mathbf{f}_b^*)$ are the applied forces statically condensed and assembled on the interface. Following a similar discussion as in the previous section, it is straightforward to show that such a splitting is equivalent to

$$\tilde{\mathbf{f}}_b^* = \mathbf{f}_b^* - \mathbf{B}_b^T\left(\mathbf{B}_b\text{diag}(\mathbf{K}_{bb})^{-1}\mathbf{B}_b^T\right)^+\mathbf{B}_b\text{diag}(\mathbf{K}_{bb})^{-1}\mathbf{f}_b^* \quad (34)$$

indicating again that the splitting corresponds to adding a particular set of equilibrated interface forces.

Let us observe that, in the non-condensed format, (32) is identical to

² This relation expresses the complementarity of the primal and dual scaling and was already suggested in [?]. Its proof can be obtained by noting that $\mathbf{L}^T(30)$ and $\mathbf{B}\mathbf{A}^{-1}(30)$ are trivially satisfied. Owing to the fact that \mathbf{A} is symmetric positive definite and because the image of \mathbf{L} corresponds to the nullspace of \mathbf{B} , the final result is obtained.

$$\begin{aligned}
\mathbf{K}\mathbf{u} + \mathbf{B}^T\boldsymbol{\lambda} &= \begin{bmatrix} \mathbf{f}_i \\ \tilde{\mathbf{f}}_b^* + \mathbf{K}_{bi}\mathbf{K}_{ii}^{-1}\mathbf{f}_i \end{bmatrix} \\
&= \mathbf{f} - \mathbf{B}^T \left(\mathbf{B} \operatorname{diag}(\mathbf{K})^{-1} \mathbf{B}^T \right)^+ \mathbf{B} \operatorname{diag}(\mathbf{K})^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_b^* \end{bmatrix}
\end{aligned} \tag{35}$$

When comparing it with (31), it is clear that splitting the condensed forces is not equivalent to splitting \mathbf{f} as commonly done. Although the displacement field obtained is obviously the same, the Lagrange multipliers searched for during the FETI iterations will be different. The question thus arises: what splitting should be used and how does it affect the FETI iterations on the dual interface problem? Next section provides a theoretical analysis of the interest of the splitting while the section after provides related numerical assessments.

4 An improved initial estimate of the Lagrange multipliers

Let us consider again the hybrid decomposed problem (12). In order to construct an estimate for the interface Lagrange multipliers, let us assume that the internal degrees of freedom \mathbf{u}_i satisfy the local equilibrium while \mathbf{u}_b on the interface boundary have a zero estimate. This is exactly the assumption underlying the initialization of the primal Schur complement iteration schemes. The subdomain equilibrium then writes

$$\mathbf{B}_b^T \boldsymbol{\lambda} \simeq \mathbf{f}_b^* \tag{36}$$

which cannot be exactly satisfied (unless $\mathbf{u}_b = 0$ corresponds to the solution). Hence, let us decompose the statically condensed force \mathbf{f}_b^* into a component satisfying the interface equilibrium (i.e. belonging to the image of \mathbf{B}_b^T) and a remainder:

$$\mathbf{B}_b^T \boldsymbol{\lambda} \simeq \mathbf{f}_b^* = \mathbf{B}_b^T \boldsymbol{\gamma} + \text{remainder} \tag{37}$$

where

$$\boldsymbol{\gamma} = \left(\mathbf{B}_b \mathbf{D} \mathbf{B}_b^T \right)^+ \mathbf{B}_b \mathbf{D} \mathbf{f}_b^* \tag{38}$$

for any non-singular and symmetric matrix \mathbf{D} . The remainder is then orthogonal to $\mathbf{D} \mathbf{B}_b^T$. If we choose as initial estimate

$$\boldsymbol{\lambda}_{00} = \boldsymbol{\gamma} \tag{39}$$

then the initial equilibrium residual $\mathbf{B}_b^T \boldsymbol{\lambda}_{00} - \mathbf{f}_b^* = \text{remainder}$ is minimum in the \mathbf{D} -norm. Setting \mathbf{D} to be $\operatorname{diag}(\mathbf{K}_{bb})^{-1}$

$$\boldsymbol{\lambda}_{00} = \left(\mathbf{B}_b \operatorname{diag}(\mathbf{K}_{bb})^{-1} \mathbf{B}_b^T \right)^+ \mathbf{B}_b \operatorname{diag}(\mathbf{K}_{bb})^{-1} \mathbf{f}_b^* \tag{40}$$

can be easily computed by a scaling of the interface condensed forces. Observing that $\text{diag}(\mathbf{K}_{bb})^{-1}$ is an approximation of \mathbf{S}^+ , $\boldsymbol{\lambda}_0$ minimizes a norm having the meaning of an energy.

We thus propose to start the FETI iterations with an initial estimate obtained by rendering $\boldsymbol{\lambda}_{00}$ admissible, i.e. such that $\mathbf{G}_I^T \boldsymbol{\lambda}_0 = \mathbf{e}$. Using the notations introduced in (17),

$$\boldsymbol{\lambda}_0 = \mathbf{P}(\mathbf{Q})\boldsymbol{\lambda}_{00} + \mathbf{Q}\mathbf{G}_I \left(\mathbf{G}_I^T \mathbf{Q}\mathbf{G}_I \right)^{-1} \mathbf{e} \quad (41)$$

To be consistent with the choice $\mathbf{D} = \text{diag}(\mathbf{K}_{bb})^{-1}$, one should take \mathbf{Q} as in (26) or (27).

Remarks

- If $\mathbf{D} = \mathbf{S}^+$ in (38), $\boldsymbol{\lambda}_{00} = \mathbf{F}_I^+ \mathbf{d}$. Furthermore, one would have $\mathbf{Q} = \mathbf{F}_I^+$ and

$$\begin{aligned} \boldsymbol{\lambda}_0 &= \mathbf{P}(\mathbf{Q})\mathbf{F}_I^+ \mathbf{d} + \mathbf{F}_I^+ \mathbf{G}_I \left(\mathbf{G}_I^T \mathbf{F}_I^+ \mathbf{G}_I \right)^{-1} \mathbf{e} \\ &= \mathbf{F}_I^+ (\mathbf{d} - \mathbf{G}_I \boldsymbol{\alpha}) \\ &\text{with } \boldsymbol{\alpha} = \left(\mathbf{G}_I^T \mathbf{F}_I^+ \mathbf{G}_I \right)^{-1} (\mathbf{G}_I^T \mathbf{F}_I^+ \mathbf{d} - \mathbf{e}) \end{aligned} \quad (42)$$

Hence the new choice of initial estimate would yield the exact solution.

- If we choose to split the interface forces such as described in (33) (or its equivalent form 34), (40) yields $\boldsymbol{\lambda}_{00} = \mathbf{0}$ so that the initial iterate (41) would correspond to the standard FETI starting procedure. Hence *it is equivalent to apply standard FETI iterations when splitting condensed forces on the interface as in (33) or to use any decomposed force vector \mathbf{f} together with the starting procedure (41)*.
- The construction of $\boldsymbol{\lambda}_{00}$ in (40) has exactly the same mechanical interpretation as the scaling of interface forces in the preconditioner (see [?]). The cost incurred by (33) or (41), corresponds to a preconditioning step, thus less than one half of a FETI iteration. Obviously, if the lumped preconditioner is applied, the standard splitting (29) should be considered.

5 Numerical assessment

In order to assess the performance of the new estimate for the Lagrange multipliers, we consider the solution to various problems by FETI and BDD. In all

cases, convergence is monitored through the evaluation of the global residual:

$$\frac{\|\mathbf{K}_g \mathbf{u}_g - \mathbf{f}_g\|}{\|\mathbf{f}_g\|} \leq \varepsilon \quad (43)$$

where \mathbf{K}_g , \mathbf{u}_g and \mathbf{f}_g are global assembled stiffness matrix, displacement field and forces (see (13)). All results are obtained with the Dirichlet preconditioner equipped with the super-lumped scaling, i.e. $\mathbf{A} = \text{diag}(\mathbf{K}_{bb})^{-1}$. The stopping criterion ε is set to 10^{-6} . The FETI method is tested for its different projectors: we note $\mathbf{P}(\tilde{\mathbf{F}}_{I_D}^{-1})$ the Dirichlet projector, $\mathbf{P}(\mathbf{W})$ the superlumped projector corresponding to (27) and $\mathbf{P}(\mathbf{I})$ the identity projector.

5.0.0.1 Cube with checkerboard heterogeneity: Let us first assess the strategies described above on the problem described in the introduction (see fig. 1). Figure 4 presents the convergence of the primal residual through the conjugate gradient iterations for the BDD and for the FETI approaches with standard force splitting. The primal approach clearly exhibit better results. Also, due to the strong heterogeneity in the structure and along the interfaces, the projector $\mathbf{P}(\mathbf{I})$ yields very poor convergence and is thus not suitable. It is observed that, although the Dirichlet projector yields a better convergence rate, the superlumped projector $\mathbf{P}(\mathbf{W})$ has a significantly lower initial residual and hence reaches convergence faster. The initialization associated to $\mathbf{P}(\mathbf{W})$ leads to a starting residual 10^3 times smaller, however its behaviour during the solution process is less regular.

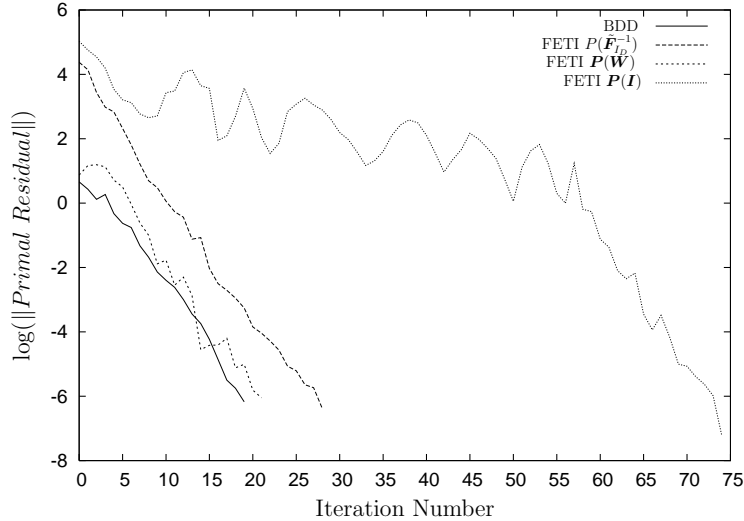


Fig. 4. Cube with checkerboard heterogeneity (fig. 1): convergence of the primal residual for FETI with standard force splitting

Figure 5 presents the convergence history for the BDD method and for the FETI method equipped with the Dirichlet projector and with the new initialization. For comparison, the convergence obtained with the standard splitting

is shown once more. Note that the new initialization not only provides a better initial estimate (lower initial residual) but also leads to similar convergence rates as for the standard splitting. Therefore the convergence history of FETI becomes very similar to the convergence of the primal BDD approach. The total number of iterations is about the same as for the primal BDD, although the convergence of FETI is less monotonic.

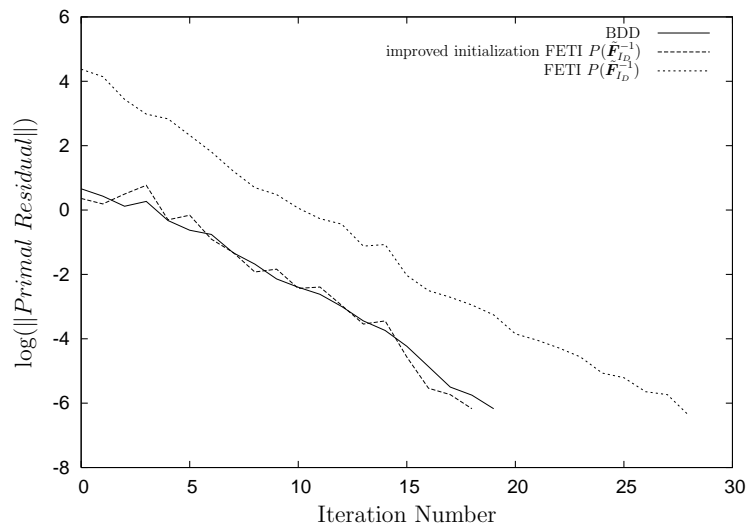


Fig. 5. Cube with checkerboard heterogeneity (fig. 1): convergence of the primal residual for FETI with new initialization

Approach		Number of iterations	Initial Residual (Log)
BDD		19	0.659
FETI Dirichlet $P(\tilde{F}_{I_D}^{-1})$	No Splitting	28	4.428
	Classical Splitting	28	4.377
	New Initialization	18	0.359
FETI Dirichlet $P(W)$	No Splitting	21	0.873
	Classical Splitting	21	0.872
	New Initialization	20	0.868
FETI Dirichlet $P(I)$	No Splitting	74	5.029
	Classical Splitting	74	5.026
	New Initialization	73	5.016

Table 1

Performance results for the cube of fig. 1

Table 1 summarizes the performance results of the various available strategies. To investigate the efficiency of the new initialization (or force splitting)

procedure, we compare the number of iterations to achieve convergence and the norm of the initial primal residual. The new initialization yields significant improvements for the Dirichlet projector (35% less iterations), but only slightly affects the convergence when other projectors are used.

5.0.0.2 Cube and slanted cube with heterogeneous layers: We now assess the new initialization strategy on two other configurations of the cube in order to evaluate the influence of the geometry and of the repartition of heterogeneities. The structures depicted in Figure 6 and 7 are similar to fig. 1 but with different material distribution. Also, for the problem described in Figure 7, the cube has been slanted by 60 degrees. In table 2 we report the number of iterations when using the BDD solver and when applying FETI with the standard and the new initialization.

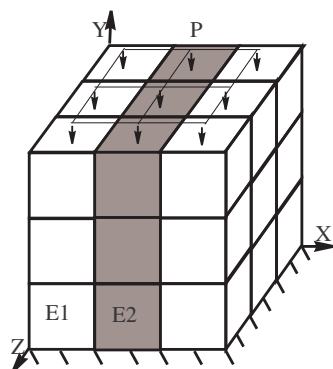


Fig. 6. Cube with heterogeneous layers

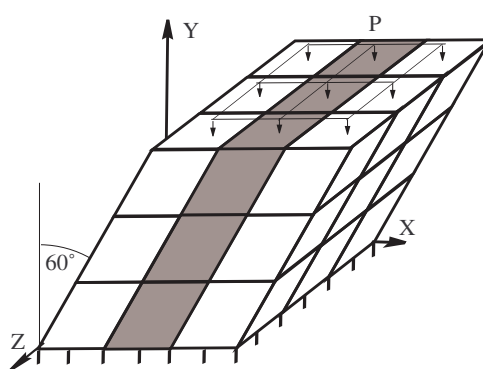


Fig. 7. Slanted cube with heterogeneous layers

For the straight cube of fig. 6, the use of the novel initialization leads to only small improvements. Observing that the FETI solver with Dirichlet projector and standard splitting converges nearly as fast as the BDD, it is clear that the new splitting strategy can not have a significant effect.

For the slanted cube, due to the geometric distortion of the substructures and of the mesh, the BDD convergence is better than when the FETI with the standard splitting is applied. When the new initialization strategy is used, the convergence history of FETI is again very similar to the convergence of the primal BDD.

5.0.0.3 Nonlinear flexion of a composite beam As a last example, we analyze a non-linear problem solved through a sequence of linearized systems. The structure (fig. 8) is a slender beam of aspect ratio 9 with square cross-section. It is made of longitudinal strips of metal and rubber. The loading corresponds to an imposed pressure on one side of the beam. Due to the presence of elastomer parts, the structure undergoes large deformations. We chose

Approach		Cube, fig. 6 Num. of it.	Slanted cube, fig. 7 Num. of it.
BDD		19	73
FETI Dirichlet $P(\tilde{F}_{I_D}^{-1})$	No Splitting	20	85
	Classical Splitting	21	85
	New Initialization	19	73
FETI Dirichlet $P(W)$	No Splitting	22	88
	Classical Splitting	22	88
	New Initialization	22	85
FETI Dirichlet $P(I)$	No Splitting	92	153
	Classical Splitting	92	154
	New Initialization	90	159

Table 2

Performance results on problem fig. 6 and 7

a Kirchoff Saint-Venant model for the metal, the characteristic coefficients of which are its Young modulus $E = 20000$ MPa and Poisson's coefficient $\nu = 0.3$. A NeoHookian model is assumed for the rubber characterized by a shear modulus $G = 2.0$ MPa and a compressibility modulus $K = 2000$ MPa. The structure is decomposed into 27 monomaterial parallelepipedic subdomains and each substructure is meshed in $2 \times 2 \times 18$ cubic elements. In order to handle the quasi-incompressibility of the elastomer, a mixed finite element formulation is considered where the pressure field and displacements are discretized independently. We choose a $Q_2 - P_1$ hexaedral element with 27 displacement nodes and 4 internal pressure nodes. Details on this formulation and on the practicalities for applying it properly in simulation can be found in [?].

The complete model has 55300 degrees of freedom of which 16400 belong to the interface. Due to the behaviour of the elastomer, the problem is highly nonlinear. Newton-Raphson iterations are performed where the tangent matrix is updated at every step. The pressure loading is 5 bars. The stopping criterion is set to 10^{-3} for the relative primal residual of FETI when solving the linearized systems. For the Newton-Raphson iterations, the tolerance for the relative residual of the nonlinear equations is set to 10^{-4} , so that 5 Newton-Raphson iterations and thus 5 linear solves must be performed.

Figure 9 indicates the number of iterations required when solving the linearized systems by the classical FETI method, by FETI with improved initialization and by the primal BDD. The Dirichlet and diagonal projectors are applied,

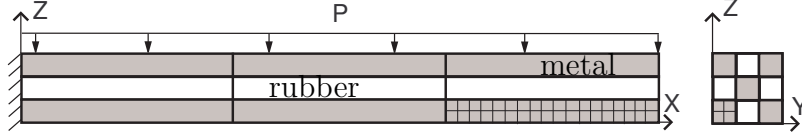


Fig. 8. $3 \times 3 \times 3$ substructures composite beam

and the Dirichlet preconditioner with stiffness scaling is used. The significant increase of the number of iterations between the first and the following linearized systems can be explained by the loss of positivity of the tangent matrix which is mostly due to incompressibility. Indeed, the conjugate gradient algorithm (with full re-orthogonalization) applied on the interface problem in the FETI and in the BDD methods remains applicable for non positive matrices but its convergence is significantly slowed down [?].³ As observed from figure 9, the new initialization enables FETI to achieve performance results which are very similar to the BDD method. Comparing the total number of iterations, the classical FETI approach requires 10% more iterations than BDD while the FETI method with new initialization requires slightly less iterations than the BDD. From figure 9 we also observe that the new initialization technique improves the performance of FETI both for the Dirichlet and the diagonal projector. For this particular structure with regular geometry, the cost effective diagonal projector yields a convergence rate very similar to the convergence rate obtained with the more computationally intensive Dirichlet projector, except for the very first linearized system solve.

Our numerical experiments indicate that for a large class of nonlinear problems such as the one depicted here, the new initialization leads to a small but non-negligible gain in terms of number of iterations and CPU time. Another important beneficial effect of the new starting procedure for FETI comes from the fact that, since the initial residual of the iterations on the interface problem are several orders of magnitude lower with the new initialization, stagnation of the residual of the FETI iterations which often happens when dealing with higher nonlinearity (higher loading) is significantly delayed, so that in practice restarting of the iteration can be avoided.

³ Often, non-positivity is due to the mutation of former null-modes (rotations) to negative modes, it can be handled by the introduction of these modes as constraints in Krylov-augmented algorithms [?]. In our case, non-positivity is mostly due to the behaviour of the rubber and the strategy described above is non-relevant. An efficient strategy based on the approximation of negative modes can be found in [?].

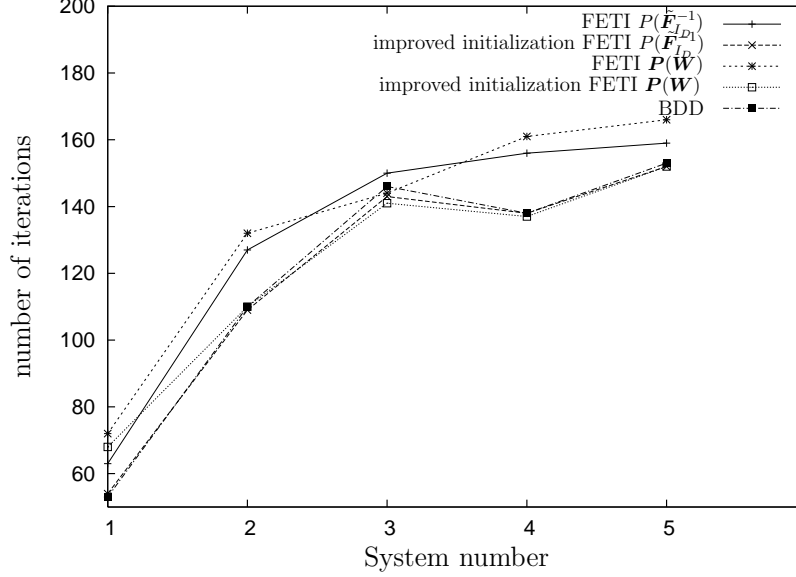


Fig. 9. Performance results for the flexion of the composite beam

6 Conclusions

For some particular structural problems such as those exhibiting strong heterogeneities and geometric distortion, the Finite Element Tearing and Interconnecting (FETI) solver can yield poor convergence compared to the conceptually similar Balanced Domain Decomposition (BDD) solver. In those cases, the bad performance of FETI can be traced back to high initial residual in the iterations on the interface problem. The initial residual is strongly related to the way the applied forces are split on the subdomain interface boundaries.

In this paper, we propose a novel strategy to split the applied forces between the subdomains. We propose to split the statically condensed interface force according to interface diagonal stiffness. This leads to building a more efficient initial estimate for the interface connecting forces. The new initialization for the FETI iterations mainly involves computing statically condensed forces on the interface and can thus be performed at a computational cost equivalent to less than half the cost of a full FETI iteration.

The numerical examples described in this paper indicate that for the problems where the primal BDD method outperforms FETI due to unexpected high initial residual, the new initialization strategy builds a better starting estimate of the interface forces and, in turn, to an initial residual similar to the BDD residual. The FETI method then converges in a manner very similar to the BDD method.

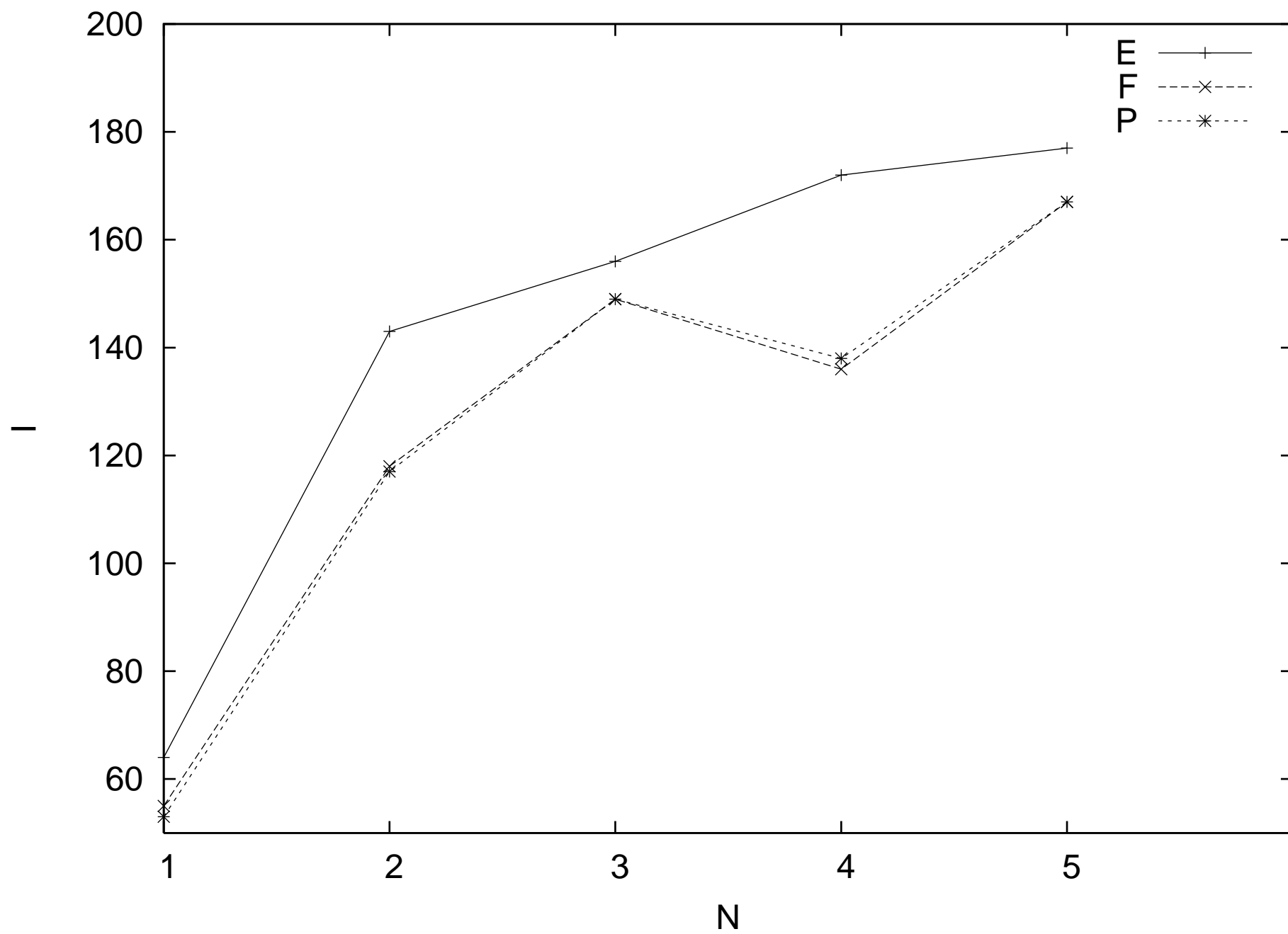
The novel starting strategy never deteriorates the FETI convergence and leads to significant improvements in some pathological cases. Therefore we suggest

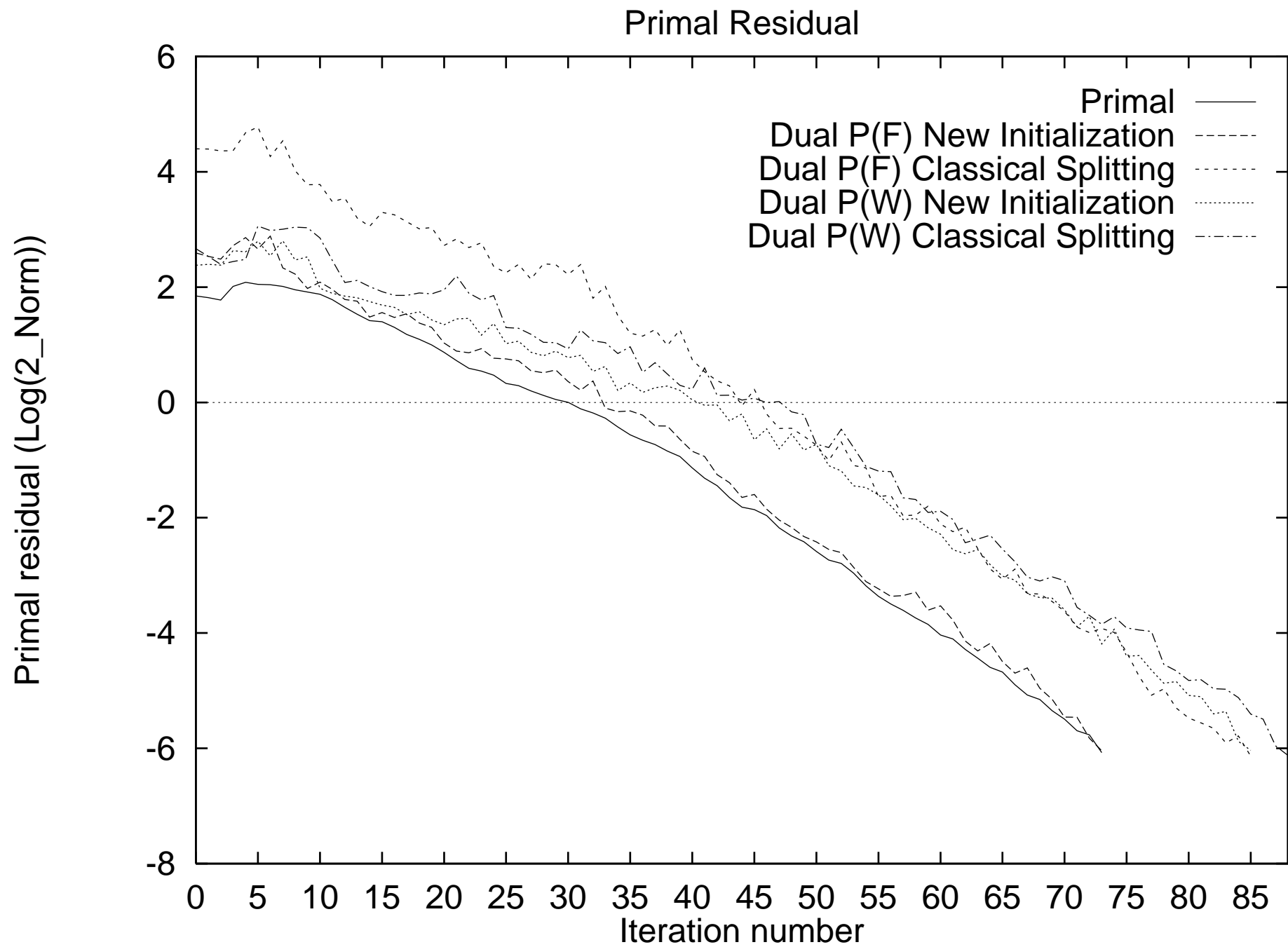
to use the presented initialization as default in FETI solvers.

With the proposed initialization for FETI, the FETI method and the BDD solver lead to similar convergence and computational costs for complex problem where the Dirichlet projector is required. For problems where the simplified FETI preconditioners and projectors can be used without significantly deteriorating the convergence of the interface iterations, FETI is often found to be more efficient in terms of overall computing cost.

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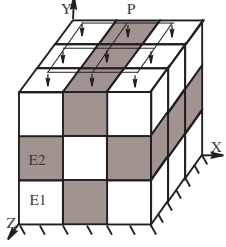


Figure 1: Decomposed heterogeneous cube

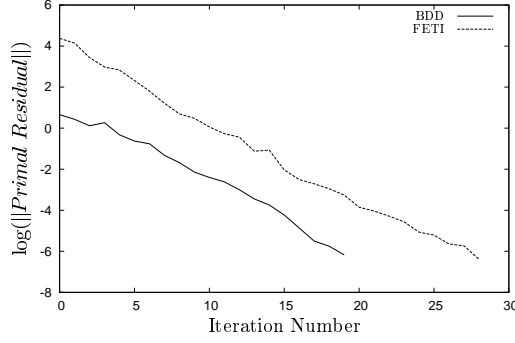


Figure 2: Convergence of BDD & FETI

an efficient way to define such a splitting and show how it is related to the construction of the initial iterate of FETI in section 4. Numerical examples are reported in section 5 to illustrate the effectiveness of the new initialization strategy. Finally, we present some conclusions.

2 FETI basics

2.1 The decomposed problem

Let us consider a domain Ω subdivided into N_s non-overlapping subdomains $\Omega^{(s)}$ and assume that we are solving a linear static equilibrium problem on the domain. The discretized subdomain equilibrium is expressed by

$$\mathbf{K}^{(s)} \mathbf{u}^{(s)} = \mathbf{f}^{(s)} + \mathbf{g}^{(s)} \quad s = 1, \dots, N_s \quad (2)$$

where $\mathbf{K}^{(s)}$, $\mathbf{u}^{(s)}$ and $\mathbf{f}^{(s)}$ are the subdomain stiffness matrices, displacements and applied forces respectively. $\mathbf{g}^{(s)}$ are the connecting forces on the interface between subdomains. For the sake of simplicity, we assume in the following that the meshes are matching (conforming) on the interface.

The interface forces satisfy an interface equilibrium equation expressing that when assembled on the interface, the resultant is null (action-reaction):

$$\sum_{s=1}^{N_s} \mathbf{L}^{(s)T} \mathbf{g}^{(s)} = \mathbf{0} \quad (3)$$

where $\mathbf{L}^{(s)}$ is a Boolean assembly matrix. The interface connecting forces are such that the interface degrees of freedom are compatible, namely

$$\sum_{s=1}^{N_s} \mathbf{B}^{(s)} \mathbf{u}^{(s)} = \mathbf{0} \quad (4)$$

This relation expresses that for any pair $(\mathbf{u}^{(s)}, \mathbf{u}^{(r)})$ of degrees of freedom matching on the interface, $\mathbf{u}^{(s)} - \mathbf{u}^{(r)} = \mathbf{0}$. $\mathbf{B}^{(s)}$ are thus signed Boolean matrices expressing the compatibility constraints on the interface.

The equilibrium problem of domain Ω is fully described by the local equilibrium (2) and by the interface constraints (3, 4). In block diagonal

notations, it can be summarized as

$$\begin{cases} \mathbf{K}\mathbf{u} &= \mathbf{f} + \mathbf{g} \\ \mathbf{L}^T\mathbf{g} &= \mathbf{0} \\ \mathbf{B}\mathbf{u} &= \mathbf{0} \end{cases} \quad (5)$$

where \mathbf{K} is the block diagonal matrix of the local operators $\mathbf{K}^{(s)}$ and where

$$\begin{aligned} \mathbf{u} &= \begin{bmatrix} \mathbf{u}^{(1)} \\ \vdots \\ \mathbf{u}^{(N_s)} \end{bmatrix} & \mathbf{f} &= \begin{bmatrix} \mathbf{f}^{(1)} \\ \vdots \\ \mathbf{f}^{(N_s)} \end{bmatrix} & \mathbf{g} &= \begin{bmatrix} \mathbf{g}^{(1)} \\ \vdots \\ \mathbf{g}^{(N_s)} \end{bmatrix} \\ \mathbf{L}^T &= \begin{bmatrix} \mathbf{L}^{(1)T} & \dots & \mathbf{L}^{(N_s)T} \end{bmatrix} \\ \mathbf{B} &= \begin{bmatrix} \mathbf{B}^{(1)} & \dots & \mathbf{B}^{(N_s)} \end{bmatrix} \end{aligned} \quad (6)$$

Note that in this description, one set of interface displacements and one set of interface forces are defined per subdomain.

2.2 Solvers for decomposed problems

Solving (5) can be done in several ways:

- Considering (5) as a constrained equilibrium problem in terms of \mathbf{u} and \mathbf{g} leads to the three-field formulation of decomposed domains (see e.g. [PJF97, RFTM99]).
- One can choose to work with a displacement set \mathbf{u} that satisfies a priori the interface compatibility (4). For that purpose we define a global set \mathbf{u}_g of degrees of freedom unique on the interface such that

$$\mathbf{u}^{(s)} = \mathbf{L}^{(s)}\mathbf{u}_g \quad \text{or} \quad \mathbf{u} = \mathbf{L}\mathbf{u}_g \quad (7)$$

where $\mathbf{L}^{(s)}$ is the same assembly Boolean matrix as in (3) that relates subdomain degrees of freedom to the global set. Stating that $\mathbf{u}^{(s)}$ are obtained from a unique set is obviously equivalent to stating the interface compatibility (4) and (7) thus implies

$$\mathbf{B}\mathbf{u} = \mathbf{B}\mathbf{L}\mathbf{u}_g = \mathbf{0} \quad (8)$$

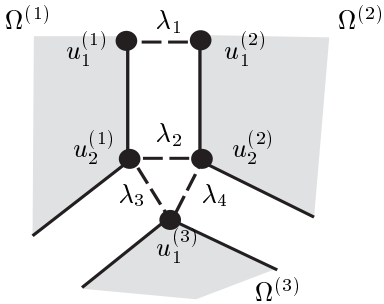
for any global displacement \mathbf{u}_g . On the other hand, all compatible displacements can be written as in (7). Hence

$$\mathbf{L} = \text{null}(\mathbf{B}) \quad (9)$$

In order to illustrate these concepts, we reproduce the example of [Rix02a] in Figure 3. Introducing (7) in (5) yields

$$\begin{cases} \mathbf{K}\mathbf{L}\mathbf{u}_g &= \mathbf{f} + \mathbf{g} \\ \mathbf{L}^T\mathbf{g} &= \mathbf{0} \end{cases} \quad (10)$$

This set of equations is at the basis of the primal iterative solution techniques such as the Primal Schur Complement or the BDD methods [TRV91, Man93]: iteration schemes are applied to find the displacements \mathbf{u}_g until the interface equilibrium $\mathbf{L}^T\mathbf{g} = \mathbf{L}^T(\mathbf{K}\mathbf{L}\mathbf{u}_g - \mathbf{f}) = \mathbf{0}$ is satisfied.



$$\underbrace{\begin{bmatrix} \overbrace{-1 \quad 0}^{B^{(1)}} & \overbrace{1 \quad 0}^{B^{(2)}} & \overbrace{0}^{B^{(3)}} \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}}_B \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}}_L \begin{bmatrix} u_{g1} \\ u_{g2} \end{bmatrix}$$

Figure 3: Lagrange multipliers and interface compatibility (from [Rix02a])

- One can choose in (5) a set of interface forces satisfying a priori the interface equilibrium $\mathbf{L}^T \mathbf{g} = \mathbf{0}$ while keeping redundant interface degrees of freedom in \mathbf{u} . According to (9), such interface forces have the generic expression

$$\mathbf{g}^{(s)} = -\mathbf{B}^{(s)T} \boldsymbol{\lambda} \quad \text{or} \quad \mathbf{g} = -\mathbf{B}^T \boldsymbol{\lambda} \quad (11)$$

$\boldsymbol{\lambda}$ are interface forces that act in opposite directions between any pair of matching degrees of freedom on the interface and are therefore in equilibrium (see Figure 3). Problem (5) becomes

$$\begin{cases} \mathbf{K}\mathbf{u} + \mathbf{B}^T \boldsymbol{\lambda} &= \mathbf{f} \\ \mathbf{B}\mathbf{u} &= \mathbf{0} \end{cases} \quad (12)$$

Clearly, $\boldsymbol{\lambda}$ are the Lagrange multipliers associated to the interface compatibility constraints. This form of the decomposed problem is the basis for the dual procedures such as FETI: iterative algorithms are applied to compute the interface forces $\boldsymbol{\lambda}$ such that the displacements resulting from the subdomain equilibrium are compatible on the interface.

- If one chooses interface displacements that are unique on part of the interface while, on the remainder of the interface, equilibrated connecting forces are defined, one obtains hybrid primal/dual approaches such as the FETI-DP procedure [FLL⁺01].
- If on the entire interface we use displacements and forces that satisfy a linear combination of the interface equilibrium and compatibility (e.g. Robin type of boundary conditions), one obtains formulations typically used in wave propagation analysis such as described for instance by Helmholtz equations [dLBFM⁺98].
- Finally if both the interface equilibrium and compatibility are enforced a priori, one obtains the fully assembled form

$$\mathbf{L}^T \mathbf{K} \mathbf{L} \mathbf{u}_g = \mathbf{L}^T \mathbf{f} \quad (13)$$

Let us note that in the case where interfaces are non-matching, the relations above remain valid, but the interface assembly and constraint matrices (\mathbf{L} and \mathbf{B}) are no longer Boolean (see for instance [BMP89]).

2.3 FETI: the dual iterative solver

In the FETI method [FR94], the decomposed problem (12) is expressed in terms of interface forces $\boldsymbol{\lambda}$: using the subdomain equilibrium equations to eliminate $\mathbf{u}^{(s)}$,

$$\mathbf{u}^{(s)} = \mathbf{K}^{(s)+} \left(\mathbf{f}^{(s)} - \mathbf{B}^{(s)T} \boldsymbol{\lambda} \right) - \mathbf{R}^{(s)} \boldsymbol{\alpha}^{(s)} \quad (14)$$

where $\mathbf{K}^{(s)+}$ is the inverse of $\mathbf{K}^{(s)}$ or a generalized inverse if subdomain $\Omega^{(s)}$ is floating when disconnected from its neighbors. In the latter case, $\mathbf{R}^{(s)}$ are the associated rigid body modes, their amplitudes $\boldsymbol{\alpha}^{(s)}$ being determined such that the interface forces are in equilibrium with the applied forces $\mathbf{f}^{(s)}$, i.e. such that the subdomain equilibrium is well-posed:

$$\mathbf{R}^{(s)T} \left(\mathbf{f}^{(s)} - \mathbf{B}^{(s)T} \boldsymbol{\lambda} \right) = \mathbf{0} \quad (15)$$